QUANTIZATION OF COBOUNDARY LIE BIALGEBRAS

BENJAMIN ENRIQUEZ AND GILLES HALBOUT

Abstract. We show that any coboundary Lie bialgebra can be quantized. For this, we prove that: (a) Etingof-Kazhdan quantization functors are compatible with Lie bialgebra twists, and (b) if such a quantization functor corresponds to an even associator, then it is also compatible with the operation of taking coopposites. We also use the relation between the Etingof-Kazhdan construction of quantization functors and the alternative approach to this problem, which was established in a previous work.

Let $\mathbf{k}$ be a field of characteristic 0. Unless specified otherwise, "algebra", "vector space", etc., means "algebra over $\mathbf{k}$", etc.

Introduction

In this paper, we solve the problem of quantization of coboundary Lie bialgebras. This is one of the quantization problems of Drinfeld's list ([Dr4]). This result can be viewed as a completion of the result of twist quantization of Lie bialgebras ([H], solving a problem posed in [KPST].)

We show that our result, together with a proposition of [Dr2], implies that quasi-Poisson manifolds over a pair $(\mathfrak{g}, Z)$ ($\mathfrak{g}$ a Lie algebra, $Z \in \wedge^3(\mathfrak{g})^{\mathfrak{g}}$) can be quantized in the case when the underlying space is the group itself (this problem was posed in [EE]).

To solve the problem of quantization of coboundary Lie bialgebras, we show that quantization functors of Lie bialgebras are compatible with Lie bialgebra twists. The quantization of all the affine Poisson groups ([DS]) of Dazord and Sondaz (i.e., Poisson homogeneous spaces under a Poisson-Lie group, which are principal as homogeneous spaces; see [Dr5]) follows immediately from this result. It is also a basic case of the quantization problem of quasi-Lie bialgebras (together with their twists) into quasi-Hopf algebras (also a problem of Drinfeld's list), which is still open.

We now describe the problem of quantization of coboundary Lie bialgebras. A coboundary Lie bialgebra is a pair $(\mathfrak{a}, r_{\mathfrak{a}})$, where $\mathfrak{a}$ is a Lie algebra (with Lie bracket denoted by $\mu_{\mathfrak{a}}$) and $r_{\mathfrak{a}} \in \wedge^2(\mathfrak{a})$ is such that $Z_{\mathfrak{a}} := [r_{\mathfrak{a}}^{12}, r_{\mathfrak{a}}^{13}] + [r_{\mathfrak{a}}^{12}, r_{\mathfrak{a}}^{23}] + [r_{\mathfrak{a}}^{13}, r_{\mathfrak{a}}^{23}] \in \wedge^3(\mathfrak{a})^{\mathfrak{a}}$. To $(\mathfrak{a}, r_{\mathfrak{a}})$ is associated a Lie bialgebra with cobracket $\delta_{\mathfrak{a}} : \mathfrak{a} \to \wedge^2(\mathfrak{a})$ given by $\delta_{\mathfrak{a}}(x) = [r_{\mathfrak{a}}, x \otimes 1 + 1 \otimes x]$.

A coboundary QUE algebra is a pair (U, R_U), where $(U, m_U, \Delta_U, \varepsilon_U, \eta_U)$ is a QUE (quantized universal enveloping) algebra (i.e., a deformation of an enveloping algebra in the category of topological $\mathbf{k}[[\hbar]]$-modules), and $R_U \in (U^{\otimes 2})^{\times}$ is such that

$$\Delta_U(x)^{21} = R_U \Delta_U(x) R_U^{-1}, \quad R_U R_U^{21} = 1_U^{\otimes 2} \tag{1}$$

$$R_U^{12}(\Delta_U \otimes \mathrm{id}_U)(R_U) = R_U^{23}(\mathrm{id}_U \otimes \Delta_U)(R_U), \tag{2}$$

$$R_U = 1_U^{\otimes 2} \mod \hbar, \quad (\varepsilon_U \otimes \mathrm{id}_U)(R_U) = (\mathrm{id}_U \otimes \varepsilon_U)(R_U) = 1_U. \tag{3}$$

(U, R_U) is a quantization of $(\mathfrak{a}, r_{\mathfrak{a}})$ if the classical limit of U is $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$, and if

$$\big(\hbar^{-1}(R_U^{21} - R_U) \mod \hbar\big) = 2r_{\mathfrak{a}} \tag{4}$$

($1_U = \eta_U(1)$ is the unit of U). The problem of quantization of coboundary Lie bialgebras is that of constructing a quantization (U, R_U) for each coboundary Lie bialgebra $(\mathfrak{a}, r_{\mathfrak{a}})$ ([Dr4, Dr1]).

Our solution is formulated in the language of props ([McL]). Recall that to a prop P and a symmetric tensor category $\mathcal{S}$, one associates the category $\mathrm{Rep}_{\mathcal{S}}(P)$ of P-modules in $\mathcal{S}$. A prop morphism $P \to Q$ then gives rise to a functor $\mathrm{Rep}_{\mathcal{S}}(Q) \to \mathrm{Rep}_{\mathcal{S}}(P)$. A quantization problem may often be formulated as the problem of constructing a functor $\mathrm{Rep}_{\mathcal{S}}(P_{\text{class}}) \to \mathrm{Rep}_{\mathcal{S}}(P_{\text{quant}})$, where $\mathcal{S} = \mathrm{Vect}$ (the category of vector spaces) and P_{class}, P_{quant} are suitable "classical" and "quantum" props. The propic version of the quantization problem is then to construct a suitable prop morphism $P_{\text{quant}} \to P_{\text{class}}$.

We construct props COB and Cob of coboundary bialgebras and of coboundary Lie bialgebras. Using an even associator defined over $\mathbf{k}$ (see [Dr3, BN]), we construct a prop morphism $\mathrm{COB} \to S(\mathbf{Cob})$ with suitable properties ($\mathbf{Cob}$ is a completion of Cob, and S is the symmetric algebra Schur functor). This allows to also solve the problem of quantization of coboundary Lie bialgebras in symmetric tensor categories (when $\mathcal{S} = \mathrm{Vect}$, this is the original problem).

Our construction is based on the theory of twists of Lie bialgebras ([Dr2]). Recall that if $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ is a Lie bialgebra, then $f_{\mathfrak{a}} \in \wedge^2(\mathfrak{a})$ is called a twist of $\mathfrak{a}$ if $(\delta_{\mathfrak{a}} \otimes \mathrm{id}_{\mathfrak{a}})(f_{\mathfrak{a}}) + [f_{\mathfrak{a}}^{13}, f_{\mathfrak{a}}^{23}]+$ cyclic permutations $= 0$. If we set $\mathrm{ad}(f_{\mathfrak{a}})(x) = [f_{\mathfrak{a}}, x^1 + x^2]$, then $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}} + \mathrm{ad}(f_{\mathfrak{a}}))$ is again a Lie bialgebra (the twisted Lie bialgebra).

A quantization of $(\mathfrak{a}, f_{\mathfrak{a}})$ is a pair (U, F_U), where $(U, m_U, \Delta_U, \varepsilon_U, \eta_U)$ is a QUE algebra quantizing $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$, and $F_U \in (U^{\otimes 2})^{\times}$ satisfies the above conditions (2), (3), and (4), with $(-2r_{\mathfrak{a}}, R_U)$ replaced by $(f_{\mathfrak{a}}, F_U)$. Then $(U, m_U, \mathrm{Ad}(F_U) \circ \Delta_U, \varepsilon_U, \eta_U)$ is again a QUE algebra (the twisted QUE algebra, denoted ${}^{F_U}U$) and is a quantization of $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}} + \mathrm{ad}(f_{\mathfrak{a}}))$ (here $\mathrm{Ad}(F_U) \in \mathrm{Aut}(U^{\otimes 2})$ is $x \mapsto F_U x F_U^{-1}$).

We notice that if $(\mathfrak{a}, r_{\mathfrak{a}})$ is a coboundary Lie bialgebra, then $-2r_{\mathfrak{a}}$ is a twist of $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}} = \mathrm{ad}(r_{\mathfrak{a}}))$, and the resulting twisted Lie bialgebra is $(\mathfrak{a}, \mu_{\mathfrak{a}}, -\delta_{\mathfrak{a}})$ (which is the coopposite of $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$). Moreover, a quantization of $(\mathfrak{a}, r_{\mathfrak{a}})$ is the same as a quantization $(U, m_U, \Delta_U, \varepsilon_U, \eta_U)$ of $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_a)$, together with a twist R_U of this QUE algebra, satisfying the additional condition (1); the second part of (1) means in particular that the twisted QUE algebra is $(U, m_U, \Delta_U^{21}, \varepsilon_U, \eta_U)$, i.e., the coopposite of the initial QUE algebra.

On the other hand, Etingof and Kazhdan constructed a quantization functor Q : Bialg $\to$ S($\mathbf{LBA}$) for each Drinfeld associator defined over $\mathbf{k}$ ([EK1]); here Bialg is the prop of bialgebras and $\mathbf{LBA}$ is a suitable completion of the prop LBA of Lie bialgebras. We also denote by Q : {Lie bialgebras over Vect} $\to$ {QUE algebras over Vect} the functor induced by this prop morphism. Our construction involves three steps:

(a) we show that any Etingof-Kazhdan quantization functor Q is compatible with twists. This is a propic version of the statement that for any $(\mathfrak{a}, f_{\mathfrak{a}})$, where $\mathfrak{a}$ is a Lie bialgebra and $f_{\mathfrak{a}}$ is a twist of $\mathfrak{a}$, there exists an element $\mathrm{F}(\mathfrak{a}, f_{\mathfrak{a}}) \in Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\otimes 2}$ satisfying the twist conditions, such that the twisted QUE algebra ${}^{\mathrm{F}(\mathfrak{a}, f_{\mathfrak{a}})}Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ is isomorphic to $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}} + \mathrm{ad}(f_{\mathfrak{a}}))$;

(b) we show that if Q corresponds to an even associator, then Q is compatible with the operation of taking coopposite Lie bialgebras and QUE algebras. This is a propic version of the statement that for any Lie bialgebra $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$, the QUE algebras $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, -\delta_{\mathfrak{a}})$ and $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{cop}}$ are isomorphic (here U^{cop} is the coopposite QUE algebra of a QUE algebra U);

(c) we are then in the following situation (at the propic level). If $(\mathfrak{a}, r_{\mathfrak{a}})$ is a coboundary Lie bialgebra, then $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, -\delta_{\mathfrak{a}}) \simeq Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{cop}}$ (where cop means the bialgebra with the opposite coproduct, and $\simeq$ is an isomorphism of QUE algebras) and $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, -\delta_{\mathfrak{a}}) \simeq Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{F}(\mathfrak{a}, -2r_{\mathfrak{a}})}$, therefore $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{cop}} \simeq Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{F}(\mathfrak{a}, -2r_{\mathfrak{a}})}$. One then proves (at the propic level) that this implies the existence of a twist $R(\mathfrak{a}, r_{\mathfrak{a}})$, such that $R(\mathfrak{a}, r_{\mathfrak{a}})R(\mathfrak{a}, r_{\mathfrak{a}})^{21} = 1_U^{\otimes 2}$ and $Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{\mathrm{cop}} = Q(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})^{R(\mathfrak{a}, r_{\mathfrak{a}})}$. This solves the quantization problem of coboundary Lie bialgebras.

Let us now describe the contents of the paper. In Section 1, we recall the formalism of props. We introduce related notions: quasi-props and quasi-bi-multiprops. Recall that a prop

(e.g., LBA) consists of the universal versions $\mathrm{LBA}(F,G)$ of the spaces of linear maps $F(\mathfrak{a}) \to G(\mathfrak{a})$ (where $\mathfrak{a}$ is a Lie bialgebra, and F,G are Schur functors), constructed from $\mu_{\mathfrak{a}}, \delta_{\mathfrak{a}}$ and avoiding cycles. The corresponding quasi-biprop consists of universal versions of the spaces of maps $F(\mathfrak{a}) \otimes F'(\mathfrak{a}^*) \to G(\mathfrak{a}) \otimes G'(\mathfrak{a}^*)$; by partial transposition, this identifies with $\mathrm{LBA}(F \otimes (G')^*, F' \otimes G^*)$, but due to the possible introduction of cycles, the composition is only partially defined: it is defined iff the "trace" of some element is. One then constructs a (partially defined) trace map $\mathrm{LBA}(F \otimes G, F \otimes G') \to \mathrm{LBA}(G, G')$, which consists in closing the graph by connecting F with itself. One can also encounter the following situation: $F = \otimes_{i=1}^n F_i$, and $x \in \mathrm{LBA}(F \otimes G, F \otimes G')$ is such that the element obtained by connecting each F_i with itself has no cycle. This defines a trace map $\mathrm{LBA}(F \otimes G, F \otimes G') \to \mathrm{LBA}(G, G')$, which depends on the data of $(F_i)_{i=1,\ldots,n}$ such that $F = \otimes_{i=1}^n F_i$; it actually depends on the multi-Schur functor $\boxtimes_{i=1}^n F_i$. In the corresponding notion of a prop (quasi-bi-multiprops), the basic objects are bi-multi-Schur functors (the "bi" analogue of a multi-Schur functor). We introduce in the end of Section 1 the main quasi-bi-multiprops we will be working with, Π and Π_f and their variants. In Section 2, we introduce the universal algebras $\mathbf{U}_n$ and $\mathbf{U}_{n,f}$ (we have morphisms $\mathbf{U}_n \to U(\mathfrak{a})^{\otimes n}$ if $\mathfrak{a}$ is any Lie bialgebra, and $\mathbf{U}_{n,f} \to U(\mathfrak{a})^{\otimes n}$ if $\mathfrak{a}$ is any Lie bialgebra equipped with a Lie bialgebra twist). In Section 3, we prove the injectivity of a map; this will be crucial for proving the compatibility of quantization functors with twists (step (a) above). In Section 4, we present the construction of quantization functors of [Enr3] (in the framework of quasi-bi-multiprops), which can be viewed as an alternative to the construction of [EK1]. Its basic ingredients are a twist J killing an associator Φ, and a factorization result for the corresponding R-matrix. In Section 5, we prove the compatibility of quantization with twists (step (a) in the above description). As in [Enr3], the proof involves two steps: an "easy" co-Hochschild cohomology argument, and a more involved injectivity result (which was proved in Section 3). In Section 6, we perform steps (b) and (c), i.e., we study the behavior of quantization functors with the operation of taking coopposites, and "correct" the twist $\mathrm{F}(\mathfrak{a}, -2r_{\mathfrak{a}})$ into a quantization of coboundary Lie bialgebras. Finally, in Subsection 6.4, we show how quantization of coboundary Lie bialgebra implies that of certain quasi-Poisson homogeneous spaces.

Notation. If $A = \oplus_{n\geq 0} A_n$ is a graded vector space, we denote by $\widehat{A} = \widehat{\oplus}_{n\geq 0} A_n$ its completion w.r.t. the grading. If A is an algebra, we denote by $A^\times$ the group of its invertible elements. If the algebra A is equipped with a character χ, then we denote by $A_1^\times$ the kernel of $\chi : A^\times \to \mathbf{k}^\times$. If A is a graded and connected algebra, then a graded character χ is unique; we will use it for defining $A_1^\times$ and $\widehat{A}_1^\times$.

1. Props and (quasi)(multi)(bi)props

In this section, we define various "Schur categories", which are all symmetric monoidal categories. We then define monoidal quasi-categories and show how they can be constructed using partial traces on monoidal categories. We then define (quasi)(multi)(bi)props, and show that variants of the prop of Lie bialgebras yield examples of these structures.

1.1. **Schur categories.** If $\mathcal{O}$ is a category, we denote by $\mathrm{Ob}(\mathcal{O})$ its set of objects and by $\mathrm{Irr}(\mathcal{O})$ the set of isomorphism classes of irreducible objects of $\mathcal{O}$. We denote by Vect the category of finite dimensional $\mathbf{k}$-vector spaces.

For $n \geq 0$, let $\widehat{\mathfrak{S}}_n$ denote the set of isomorphism classes of irreductible representations of $\mathfrak{S}_n$ (by convention, $\mathfrak{S}_0 = \{1\}$). We view $\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$ as the set of pairs (n,π), where $n \geq 0$ and $\pi \in \widehat{\mathfrak{S}}_n$. For $\rho = (n,\pi)$, we set $|\rho| := n$ and $\pi_\rho := \pi$, so $\rho = (|\rho|, \pi_\rho)$. If σ, τ are finite dimensional representations of $\mathfrak{S}_n$, $\mathfrak{S}_m$, then $\sigma * \tau$ is defined as $\mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(\sigma \otimes \tau)$; we have then an identification $(\sigma * \sigma') * \sigma'' \simeq \sigma * (\sigma' * \sigma'')$. The dual representation of ρ is denoted ρ^*.

1.1.1. *The category* Sch. Define the Schur category Sch as follows. $\mathrm{Ob}(\mathrm{Sch}) := \mathrm{Ob}(\mathrm{Vect})^{(\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n)} =$ {finitely supported families $F = (F_\rho)$ of finite dimensional vector spaces, indexed by $\rho \in \sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$}. For $F = (F_\rho)$ and $G = (G_\rho)$ in Ob(Sch), we set $\mathrm{Sch}(F,G) := \oplus_\rho \mathrm{Vect}(F_\rho, G_\rho)$; $F \oplus G = (F_\rho \oplus G_\rho)$; $F^* := (F^*_{\rho^*})$; and $(F\otimes G)_\rho := \oplus_{\rho',\rho''\in\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n} F_{\rho'} \otimes G_{\rho''} \otimes \mu^\rho_{\rho'\rho''}$, where for $\rho,\rho',\rho'' \in \sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$, we set $\mu^\rho_{\rho'\rho''} = \mathrm{Hom}_{\mathfrak{S}_{|\rho|}}(\pi_{\rho'} * \pi_{\rho''}, \pi_\rho)$ if $|\rho| = |\rho'| + |\rho''|$ and 0 otherwise. The direct sum, and the involution $\rho \mapsto \rho^*$ followed by the transposition induce canonical maps $\mathrm{Sch}(F,G)\oplus\mathrm{Sch}(F',G') \to \mathrm{Sch}(F\oplus F', G\oplus G')$ and $\mathrm{Sch}(F,G) \to \mathrm{Sch}(G^*,F^*)$, and for $f = (f_\rho) \in \mathrm{Sch}(F,F')$ and $g = (g_\rho) \in \mathrm{Sch}(G,G')$, we define $(f\otimes g)_\rho := \oplus_{\rho',\rho''\in\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n} f_{\rho'} \otimes g_{\rho''} \otimes \mathrm{id}_{\mu^\rho_{\rho'\rho''}}$.

Then Sch is a symmetric additive strict monoidal category with an anti-automorphism[1]; it is also Karoubian (i.e., every projector has a kernel and a cokernel). We have a canonical bijection $\mathrm{Irr}(\mathrm{Sch}) \simeq \sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$, with inverse given by $\rho \mapsto Z_\rho$, where $(Z_\rho)_{\rho'} = \mathbf{k}$ if $\rho' = \rho$ and 0 otherwise. We denote by $\mathbf{1}$, $\mathbf{id}$, S^n, $\wedge^n$ the elements of Irr(Sch) corresponding to the elements of $\widehat{\mathfrak{S}}_0$, $\widehat{\mathfrak{S}}_1$, the trivial and the signature characters of $\mathfrak{S}_n$; $\mathbf{1}$ is the unit object of Sch. Sch has the following universal property: if $\mathcal{C}$ is a Karoubian additive symmetric strict monoidal category with a distinguished object M, then there exists a unique tensor functor $F_{(\mathcal{C},M)} : \mathrm{Sch} \to \mathcal{C}$ such that $F(\mathbf{id}) = M$. In particular, for $G \in \mathrm{Ob}(\mathrm{Sch})$, we get an endofunctor $F_{(\mathrm{Sch},G)} : \mathrm{Sch} \to \mathrm{Sch}$, which we denote by $F \mapsto F\circ G$ (or $F(G)$) at the level of objects and $f \mapsto f \circ G$ (or $f(G)$) at the level of morphisms. We say that $F = (F_\rho) \in \mathrm{Ob}(\mathrm{Sch})$ is homogeneous of degree n iff $F_\rho = 0$ for $|\rho| \neq n$. If F is a homogeneous Schur functor, we denote by $|F|$ its degree.

Let End(Vect) be the symmetric additive strict monoidal category where objects are endofunctors $F : \mathrm{Vect} \to \mathrm{Vect}$, and morphisms $F \to G$ are natural transformations, i.e., assignments $\mathrm{Vect} \ni V \mapsto f_V \in \mathrm{Vect}(F(V), G(V))$, such that $f_W \circ F(\phi) = G(\phi)\circ f_V$ for $\phi \in \mathrm{Vect}(V,W)$. We define direct sums in End(Vect) by $(F \oplus F')(V) := F(V) \oplus F'(V)$ and $(V \mapsto f_V) \oplus (V \mapsto f'_V) := (V \mapsto f_V \oplus f'_V)$. We define a tensor product in End(Vect) by $(F\otimes F')(V) := F(V)\otimes F'(V)$ and $(V \mapsto f_V)\otimes(V\mapsto f'_V) := (V \mapsto f_V \otimes f'_V)$. We define an anti-automorphism of End(Vect) by $F^*(V) := F(V^*)^*$ and $(V \mapsto f_V)^* := (V \mapsto f^*_{V^*})$, where $(-)^*$ is the transposed endomorphism. Each $G \in \mathrm{End(Vect)}$ gives rise to an endomorphism of End(Sch), $F \mapsto F\circ G$, where $F \circ G(V) := F(G(V))$.

We then have a tensor functor $\mathrm{Sch} \to \mathrm{End(Vect)}$, compatible with the anti-automorphisms and with the endomorphisms $F \mapsto F\circ G$, defined at the level of objects by $F = (F_\rho) \mapsto (V \mapsto \oplus_{\rho\in\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n} F_\rho \otimes Z_\rho(V)$, where $Z_\rho(V) := \mathrm{Hom}_{\mathfrak{S}_{|\rho|}}(\pi_\rho, V^{\otimes|\rho|})$, and at the level of morphisms by $f = (f_\rho) \mapsto (V \mapsto f_V)$, where $f_V = \oplus_{\rho\in\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n} f_\rho \otimes \mathrm{id}_{Z_\rho(V)}$.

For later use, we define[2] the set $\mathrm{Ob}(\mathrm{Sch}_k) := \mathrm{Ob}(\mathrm{Vect})^{((\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n)^k)} =$ {finitely supported families $F = (F_{\rho_1,\ldots,\rho_k})$ of finite dimensional vector spaces, indexed by $(\rho_1,...,\rho_k) \in (\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n)^k$}. The direct sums and duality are defined component-wise as before. Note that $\mathrm{Ob}(\mathrm{Sch}_0) = \mathrm{Ob}(\mathrm{Vect})$. If[3] $\phi : [m] \to [n]$ is a partially defined map (often identified with the collection of preimages $\phi^{-1}(1),...,\phi^{-1}(n)$), we define an additive map $\Delta^\phi : \mathrm{Ob}(\mathrm{Sch}_n) \to \mathrm{Ob}(\mathrm{Sch}_m)$, taking $(F_{\rho_1,\ldots,\rho_n})$ to $(\Delta^\phi(F)_{\pi_1,\ldots,\pi_m})$, where $\Delta^\phi(F)_{\pi_1,\ldots,\pi_m} = \oplus_{\rho_1,\ldots,\rho_m}(\mu^{\rho_1}_{\pi_i, i\in\phi^{-1}(1)})^* \otimes ... \otimes (\mu^{\rho_n}_{\pi_i,i\in\phi^{-1}(n)})^*) \otimes F_{\rho_1\ldots\rho_n}$. We set $\Delta := \Delta^{\{1,2\}} : \mathrm{Ob}(\mathrm{Sch}) \to \mathrm{Ob}(\mathrm{Sch}_2)$. Let $\mathrm{Fun}(\mathrm{Vect}^n, \mathrm{Vect})$ be the set of functors $\mathrm{Vect}^n \to \mathrm{Vect}$; the direct sum is defined by $(F \oplus G)(V_1,...,V_n) := F(V_1,...,V_n) \oplus G(V_1,...,V_n)$ and the duality by $F^*(V_1,...,V_n) := F(V_1^*,...,V_n^*)^*$; we also define

[1] An anti-automorphism of a category $\mathcal{C}$ is the data of a permutation $X \mapsto X^*$ of $\mathrm{Ob}(\mathcal{C})$, and of maps $\mathcal{C}(X,Y) \to \mathcal{C}(Y^*,X^*)$, $x \mapsto x^*$, such that $(y\circ x)^* = x^*\circ y^*$; if $\mathcal{C}$ is additive, we require compatibility with direct sums and the linear structure of the $\mathcal{C}(X,Y)$; if $\mathcal{C}$ is monoidal, we require $(X\otimes Y)^* = X^*\otimes Y^*$, $\mathbf{1}^* = \mathbf{1}$ and $(x\otimes y)^* = x^*\otimes y^*$.

[2] For I a finite set, we define $\mathrm{Ob}(\mathrm{Sch}_I)$ similarly, where $(\rho_1,..,\rho_K)$ is replaced by a map $I \to \sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$.

[3] We set $[n] := \{1,...,n\}$.

$\Delta^\phi : \mathrm{Fun}(\mathrm{Vect}^n, \mathrm{Vect}) \to \mathrm{Fun}(\mathrm{Vect}^m, \mathrm{Vect})$ by $(\Delta^\phi F)(V_1, ..., V_m) := F(\oplus_{i\in\phi^{-1}(1)} V_i, ..., \oplus_{i\in\phi^{-1}(n)} V_i)$. Then the map $\mathrm{Sch}_n \to \mathrm{Fun}(\mathrm{Vect}^n, \mathrm{Vect})$ taking $(F_{\rho_1...\rho_n})$ to $F : (V_1, ..., V_n) \mapsto \oplus_{\rho_1,...,\rho_n} F_{\rho_1,...,\rho_n} \otimes Z_{\rho_1}(V_1) \otimes ... \otimes Z_{\rho_n}(V_n)$ is compatible with the direct sums, the duality and the maps Δ^ϕ.

1.1.2. *The category* Sch_{1+1}. We define the symmetric additive strict monoidal category of Schur bifunctors Sch_{1+1} as follows. $\mathrm{Ob}(\mathrm{Sch}_{1+1}) := \mathrm{Ob}(\mathrm{Sch}_2)$. For $F, G \in \mathrm{Ob}(\mathrm{Sch}_{1+1})$, $\mathrm{Sch}_{1+1}(F, G) := \oplus_{\rho_1,\rho_2} \mathrm{Vect}(F_{\rho_1,\rho_2}, G_{\rho_1,\rho_2})$. We also define $(F\otimes G)_{\rho_1,\rho_2} := \oplus_{\rho'_i,\rho''_i} F_{\rho'_1,\rho'_2} \otimes G_{\rho''_1,\rho''_2} \otimes \mu^{\rho_1}_{\rho'_1\rho''_1} \otimes \mu^{\rho_2}_{\rho'_1\rho''_2}$. The direct sums and tensor products of morphisms are then defined component-wise. An anti-automorphism of Sch_{1+1} is defined by $(F_{\rho,\sigma})^* = (F^*_{\sigma^*,\rho^*})$ and by $((\rho, \sigma) \mapsto f_{\rho,\sigma})^* = ((\rho, \sigma) \mapsto f^*_{\sigma^*,\rho^*})$. We define a tensor morphism $\boxtimes : \mathrm{Sch}^2 \to \mathrm{Sch}_{1+1}$ at the level of objects by $(F \boxtimes G)_{\rho,\sigma} := F_\rho \otimes G_\sigma$, and component-wise at the level of morphisms. We then have for $F, ..., G' \in \mathrm{Ob}(\mathrm{Sch})$,

$$\mathrm{Sch}_{1+1}(F \boxtimes G^*, F' \boxtimes G'^*) \simeq \mathrm{Sch}(F, F') \otimes \mathrm{Sch}(G', G) \tag{5}$$

and $(F\boxtimes G)^* = G^*\boxtimes F^*$. As Sch_{1+1} is Karoubian, any $G \in \mathrm{Sch}_{1+1}$ gives rise to a unique tensor functor $\mathrm{Sch} \to \mathrm{Sch}_{1+1}$ taking $\mathbf{id}$ to G, which we denote by $F \mapsto F \circ G$.

Let $\mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$ be the symmetric additive strict monoidal category where objects are functors $\mathrm{Vect}^2 \to \mathrm{Vect}$ and morphisms are natural transformations; the direct sum is defined by $(F \oplus G)(V, W) := F(V, W) \oplus G(V, W)$, the tensor products by $(F \otimes G)(V, W) := F(V, W) \otimes G(V, W)$ and the duality by $F^*(V, W) := F(W^*, V^*)^*$. Then we have a tensor functor $\mathrm{Sch}_{1+1} \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$ taking F to $((V, W) \mapsto \oplus_{\rho_1,\rho_2} F_{\rho_1,\rho_2} \otimes Z_{\rho_1}(V) \otimes Z_{\rho_2}(W))$, compatible with the dualities. It is also compatible with the tensor functor $\mathrm{End}(\mathrm{Sch}) \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$, $G \mapsto F \circ G$, where $F \circ G(V, W) = F(G(V, W))$. We define a tensor functor $\boxtimes : \mathrm{End}(\mathrm{Vect})^2 \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$ at the level of objects by $(F\boxtimes G)(V, W) := F(V)\otimes G(W)$. Then the morphisms $\mathrm{Sch} \to \mathrm{End}(\mathrm{Vect})$, $\mathrm{Sch}_{1+1} \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$ intertwine the functors $\boxtimes : \mathrm{Sch}^2 \to \mathrm{Sch}_{1+1}$ and $\boxtimes : \mathrm{End}(\mathrm{Vect})^2 \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$.

1.1.3. *The category* $\mathrm{Sch}_{(1)}$. We now define the additive symmetric strict monoidal category $\mathrm{Sch}_{(1)}$ as follows. We set $\mathrm{Ob}(\mathrm{Sch}_{(1)}) := \mathrm{Ob}(\mathrm{Vect})^{(\sqcup_{k\geq 0}(\sqcup_{n\geq 0}\mathfrak{S}_n)^k)} = \prod'_{k\geq 0} \mathrm{Ob}(\mathrm{Sch}_k) = \{$finitely supported collections $(F_k)_{k\geq 0}$, where $F_k \in \mathrm{Ob}(\mathrm{Sch}_k)$ is a family $F_k = (F_{\rho_1,...,\rho_k})\}$. The direct sum of objects is defined by component-wise addition. The tensor product of objects is defined by $(F_k) \boxtimes (G_k) := ((F \boxtimes G)_k)$, where $(F \boxtimes G)_k := \oplus_{k',k''|k'+k''=k} F_{k'} \boxtimes G_{k''}$, and if $F_{k'} = (F_{\rho_1,...,\rho_{k'}}) \in \mathrm{Ob}(\mathrm{Sch}_{k'})$, $G_{k''} = (G_{\rho_1,...,\rho_{k''}}) \in \mathrm{Ob}(\mathrm{Sch}_{k''})$, then $F_{k'} \boxtimes G_{k''} = ((F \boxtimes G)_{\rho_1,...,\rho_{k'+k''}}) \in \mathrm{Ob}(\mathrm{Sch}_{k'+k''})$, where $(F \boxtimes G)_{\rho_1,...,\rho_{k'+k''}} := F_{\rho_1,...,\rho_{k'}} \otimes G_{\rho_{k'+1},...,\rho_{k'+k''}}$.

In order to define the morphisms, we first define a "contraction" map $c : \mathrm{Ob}(\mathrm{Sch}_k) \to \mathrm{Ob}(\mathrm{Sch})$, $F_k = (F_{\rho_1,...,\rho_k}) \mapsto c(F_k)$ by $c(F_k)_\rho := \oplus_{\rho_1,...,\rho_k} F_{\rho_1,...,\rho_k} \otimes \mu^\rho_{\rho_1,...,\rho_k}$, where $\mu^\rho_{\rho_1,...,\rho_k} := \mathrm{Hom}_{\mathfrak{S}_{|\rho|}}(\rho_1 * ... * \rho_k, \rho)$ if $\sum_i |\rho_i| = |\rho|$ and 0 otherwise. For $F = (F_k)$, we then set $c(F) := \oplus_k c(F_k)$ and for $F, G \in \mathrm{Ob}(\mathrm{Sch}_{(1)})$, we set $\mathrm{Sch}_{(1)}(F, G) := \mathrm{Sch}(c(F), c(G))$. We define the direct sum and the tensor product of morphisms using the identifications $c(F \oplus G) \simeq c(F) \oplus c(G)$ and $c(F \boxtimes G) \simeq c(F) \otimes c(G)$. The symmetry constraint in $\mathrm{Sch}_{(1)}(F \boxtimes G, G \boxtimes F) = \mathrm{Sch}(c(F \boxtimes G), c(G \boxtimes F))$ is then given by the identifications $c(X \boxtimes Y) \simeq c(X) \otimes c(Y)$ and the symmetry constraint for Sch. The unit object of $\mathrm{Ob}(\mathrm{Sch}_{(1)})$ is $\mathbb{1}$, whose only nonzero component is $\mathbb{1}_0 = \mathbf{k} \in \mathrm{Ob}(\mathrm{Sch}_0) = \mathrm{Ob}(\mathrm{Vect})$.

We define an additive symmetric strict monoidal category $\prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$ as follows. The objects are finitely supported families $(F_k)_{k\geq 0}$, where $F_k \in \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$. The direct sum of objects is defined component-wise, where for $F_k, G_k \in \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$, $F_k \oplus G_k \in \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$ is given by $(F_k \oplus G_k)(V_1, ..., V_k) := F_k(V_1, ..., V_k) \oplus G_k(V_1, ..., V_k)$. The tensor product of objects is $(F_k) \boxtimes (G_k) := ((F \boxtimes G)_k)$, where $(F \boxtimes G)_k := \oplus_{k'+k''=k} F_{k'} \boxtimes_{k',k''} G_{k''}$, and $\boxtimes_{k',k''} : \mathrm{Fun}(\mathrm{Vect}^{k'}, \mathrm{Vect}) \times \mathrm{Fun}(\mathrm{Vect}^{k''}, \mathrm{Vect}) \to \mathrm{Fun}(\mathrm{Vect}^{k'+k''}, \mathrm{Vect})$ is given by

$(F_{k'} \boxtimes_{k',k''} G_{k''})(V_1,...,V_{k'+k''}) := F_{k'}(V_1,...,V_{k'}) \otimes G_{k''}(V_{k'+1},...,V_{k'+k''})$. The contraction $c : \prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect}) \to \mathrm{End}(\mathrm{Fun})$ is defined by $c((F_k)) := \oplus_{k\geq 0} c(F_k)$, where $c(F_k)(V) := F_k(V,...,V)$. The space of morphisms $F \to G$ is then defined as $\mathrm{Fun}(\mathrm{Vect})(c(F), c(G))$.

There is a unique tensor morphism $\mathrm{Sch}_{(1)} \to \prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$, taking $F = (F_k)_{k\geq 0}$, where $F_k = (F_{\rho_1,...,\rho_k})$, to the collection $(\tilde F_k)_{k\geq 0}$, where $\tilde F_k(V_1,...,V_k) := \oplus_{\rho_1,...,\rho_k} F_{\rho_1,...,\rho_k} \otimes Z_{\rho_1}(V_1) \otimes ... \otimes Z_{\rho_k}(V_k)$.

1.1.4. *The category* $\mathrm{Sch}_{(1+1)}$. We now define an additive symmetric strict monoidal category $\mathrm{Sch}_{(1+1)}$ as follows. $\mathrm{Ob}(\mathrm{Sch}_{(1+1)}) := \mathrm{Ob}(\mathrm{Vect})^{(\sqcup_{k,l\geq 0}(\sqcup_{n\geq 0}\widehat{\mathfrak S}_n)^{k+l})} = \{$finitely supported collections $(F_{k,l})$, where $F_{k,l} = (F_{\rho_1,...,\rho_k;\sigma_1,...,\sigma_l}) \in \mathrm{Ob}(\mathrm{Sch}_{k+l})\}$. The direct sum of objects is defined component-wise, and the tensor product is given by $(F \boxtimes F')_{k,l} := \oplus_{(k_1,l_2)+(k_2,l_2)=(k,l)} F_{k_1,l_1} \boxtimes_{k_1,l_1,k_2,l_2} F'_{k_2,l_2}$ if $F = (F_{k,l})$ and $F' = (F'_{k,l})$, where $\boxtimes_{k,l,k',l'} : \mathrm{Ob}(\mathrm{Sch}_{k+l}) \times \mathrm{Ob}(\mathrm{Sch}_{k'+l'}) \to \mathrm{Ob}(\mathrm{Sch}_{k+k'+l+l'})$ is given by
$(F \boxtimes_{k,l,k',l'} F')_{\rho_1,...,\rho_{k+k'};\sigma_1,...,\sigma_{l+l'}} := F_{\rho_1,...,\rho_k;\sigma_1,...,\sigma_l} \otimes F'_{\rho_{k+1},...,\rho_{k+k'};\sigma_{l+1},...,\sigma_{l+l'}}$. An involution is defined by $F^* = ((k,l,\rho_1,...,\rho_k,\sigma_1,...,\sigma_l) \mapsto F(l,k,\sigma_1^*,...,\sigma_l^*,\rho_1^*,...,\rho_l^*)^*)$ for $F = ((k,l,\rho_1,...,\rho_k,\sigma_1,...,\sigma_l) \mapsto F(k,l,\rho_1,...,\rho_k,\sigma_1,...,\sigma_l))$.

In order to define the morphisms, we define a map $c : \mathrm{Ob}(\mathrm{Sch}_{(1+1)}) \to \mathrm{Ob}(\mathrm{Sch}_{1+1})$, $F \mapsto c(F)$, by $c(F) := \oplus_{k,l} c(F_{k,l})$ for $F = (F_{k,l})$, and if $F_{k,l} = (F_{\rho_1,...,\rho_k;\sigma_1,...,\sigma_l})$, then $c(F_{k,l})_{\rho,\sigma} = \oplus_{\rho_1,...,\rho_k;\sigma_1,...,\sigma_k} F_{\rho_1,...,\rho_k;\sigma_1,...,\sigma_l} \otimes \mu^\rho_{\rho_1...\rho_k} \otimes \mu^\sigma_{\sigma_1...\sigma_l}$. We then set $\mathrm{Sch}_{(1+1)}(F,G) := \mathrm{Sch}_{1+1}(c(F), c(G))$. The direct sum, tensor product and duality of morphisms are then induced by those of Sch_{1+1} and the identifications $c(F \oplus G) \simeq c(F) \oplus c(G)$, $c(F \boxtimes G) \simeq c(F) \boxtimes c(G)$, $c(F^*) = c(F)^*$.

We define a tensor morphism $\underline{\boxtimes} : (\mathrm{Sch}_{(1)})^2 \to \mathrm{Sch}_{(1+1)}$; at the level of objects, it is defined by $(F_k)\underline{\boxtimes}(G_l) := (F_k \boxtimes_{k,l} G_l)$; at the level of morphisms, it is induced by the tensor morphism $\mathrm{Sch}^2 \to \mathrm{Sch}_{1+1}$. The unit object of $\mathrm{Sch}_{(1+1)}$ is $\mathbf{1}\underline{\boxtimes}\mathbf{1}$. We then have for $F,...,G' \in \mathrm{Ob}(\mathrm{Sch}_{(1)})$

$$\mathrm{Sch}_{(1+1)}(F\underline{\boxtimes}G^*, F'\underline{\boxtimes}G'^*) = \mathrm{Sch}_{(1)}(F,F') \otimes \mathrm{Sch}_{(1)}(G',G)$$

and $(F\underline{\boxtimes}G)^* = G^*\underline{\boxtimes}F^*$.

As before, we define an additive symmetric strict monoidal category $\prod'_{k,l} \mathrm{Fun}(\mathrm{Vect}^{k+l}, \mathrm{Vect})$; objects are fiinitely supported families $(F_{k,l})_{k,l\geq 0}$; $(F_{k,l} \oplus G_{k,l})(V_1,...,V_k;W_1,...,W_l) := F_{k,l}(V_1,...,W_l) \oplus G_{k,l}(V_1,...,W_l)$; $\boxtimes_{k',l',k'',l''} : \mathrm{Fun}(\mathrm{Vect}^{k'+l'}, \mathrm{Vect}) \times \mathrm{Fun}(\mathrm{Vect}^{k''+l''}, \mathrm{Vect}) \to \mathrm{Fun}(\mathrm{Vect}^{k'+l'+k''+l''}, \mathrm{Vect})$ is $(F \boxtimes_{k',l',k'',l''} G)(V_1,...,W_{l'+l''}) := F(V_1,...,W_{l'}) \otimes G(V_{k'+1},...,W_{l'+l''})$. We define
$c : \prod'_{k,l} \mathrm{Fun}(\mathrm{Vect}^{k+l}, \mathrm{Vect}) \to \mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})$ by $c(F) = \oplus_{k,l} c(F_{k,l})$ and $c(F_{k,l})(V,W) := F_{k,l}(V,...,V;W,...,W)$ and the space of morphisms $F \to G$ as $\mathrm{Fun}(\mathrm{Vect}^2, \mathrm{Vect})(c(F), c(G))$. We also define a tensor morphism $\underline{\boxtimes} : (\prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect}))^2 \to \prod'_{k,l\geq 0} \mathrm{Fun}(\mathrm{Vect}^{k+l}, \mathrm{Vect})$ at the level of objects by $(F_k)\underline{\boxtimes}(G_k) := (F_k \boxtimes_{k,l} G_l)$.

Then we have a tensor morphism $\mathrm{Sch}_{(1+1)} \to \prod'_{k,l\geq 0} \mathrm{Fun}(\mathrm{Vect}^{k+l}, \mathrm{Vect})$, taking $(F_{k,l})$ to $(\tilde F_{k,l})$, where for $F_{k,l} = (F_{\rho_1,...,\sigma_l})$, $\tilde F_{k,l}(V_1,...,W_l) := \oplus_{\rho_1,...,\sigma_l} F_{\rho_1,...,\sigma_l} \otimes \rho_1(V_1) \otimes ... \otimes \sigma_l(W_l)$. This morphism is compatible with the morphism $\mathrm{Sch}_{(1)} \to \prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect})$ and the morphisms $(\mathrm{Sch}_{(1)})^2 \to \mathrm{Sch}_{(1+1)}$, $(\prod'_{k\geq 0} \mathrm{Fun}(\mathrm{Vect}^k, \mathrm{Vect}))^2 \to \prod'_{k,l\geq 0} \mathrm{Fun}(\mathrm{Vect}^{k+l}, \mathrm{Vect})$.

1.1.5. *Completions.* If in the definition of Sch, we forget the condition that (F_ρ) is finitely supported, we get a symmetric additive strict monoidal category with duality $\mathbf{Sch}$. Infinite sums of objects of increasing degrees are defined in $\mathbf{Sch}$. For each $G = \oplus_{i\geq 1} G_i \in \mathbf{Sch}$ ($|G_i| = i$), we have an endofunctor of $\mathbf{Sch}$, $F \mapsto F \circ G$, $f \mapsto f \circ G$ (also written $F(G), f(G)$). We also define $\mathrm{Ob}(\mathbf{Sch}_k)$ by dropping the finite support condition. The maps Δ^ϕ extend to these sets. We define $\mathbf{Sch}_{1+1}$, $\mathbf{Sch}_{(1)}$ and $\mathbf{Sch}_{(1+1)}$ similarly to Sch_{1+1}, $\mathrm{Sch}_{(1)}$, $\mathrm{Sch}_{(1+1)}$, namely $\mathrm{Ob}(\mathbf{Sch}_{1+1}) = \mathrm{Ob}(\mathbf{Sch}_2)$, $\mathrm{Ob}(\mathbf{Sch}_{(1)}) = \{$finitely supported families $(F_k)_{k\geq 0}$, where $F_k \in \mathrm{Ob}(\mathbf{Sch}_k)$, and $\mathrm{Ob}(\mathbf{Sch}_{(1+1)}) = \{$finitely supported families $(F_{k,l})_{k,l\geq 0}$, where $F_{k,l} \in \mathrm{Ob}(\mathbf{Sch}_{k+l})\}$. Then

Sch_{1+1}, $\mathrm{Sch}_{(1)}$ and $\mathrm{Sch}_{(1+1)}$ have structures of additive symmetric strict monoidal categories. The map c and the bifunctor $\boxtimes$ extend to these categories; the duality extends to $\mathbf{Sch}_{1+1}$ and $\mathbf{Sch}_{(1+1)}$.

Examples. Let $T_n \in \mathrm{Ob}(\mathrm{Sch})$ be such that $(T_n)_{\rho'} = \pi_{\rho'}$ if $|\rho'| = n$ and 0 otherwise. The corresponding endofunctor of Vect is $V \mapsto T_n(V) = V^{\otimes n} = \oplus_{\rho\in\widehat{\mathfrak{S}}_n}\pi_\rho \otimes Z_\rho(V)$. Using the obvious module category structure of Sch over Vect, we write

$$T_n = \oplus_{Z\in\mathrm{Irr(Sch)},|Z|=n}\pi_Z \otimes Z, \tag{6}$$

where $Z \mapsto (|Z|, \pi_Z)$ is the inverse to $\sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n \to \mathrm{Irr(Sch)}$, $\rho \mapsto Z_\rho$.

The endofunctors of Vect corresponding to S^n and $\wedge^n$ are the nth symmetric and exterior powers functors. The symmetric and exterior algebra functors $S := \oplus_{n\geq 0}S^n$ and $\wedge := \oplus_{n\geq 0}\wedge^n$ are objects in **Sch**. We then have $\Delta(S) = S \boxtimes S$, $\Delta(\wedge) = \wedge \boxtimes \wedge$. Note that while the map $\mathrm{Ob(Sch)} \to \mathrm{Ob(End(Vect))}$ is injective, it is not surjective, e.g. the exterior algebra functor is not in the image of this map.

Remark 1.1. For any $F, G \in \mathrm{Ob(Sch)}$, we have

$$\mathrm{Sch}(F, G) = \oplus_{Z\in\mathrm{Irr(Sch)}}\, \mathrm{Sch}(F, Z) \otimes \mathrm{Sch}(Z, G); \tag{7}$$

for any $B, B' \in \mathrm{Ob}(\mathrm{Sch}_{1+1})$, we have

$$\mathrm{Sch}_{1+1}(B, B') = \oplus_{Z,Z'\in\mathrm{Irr(Sch)}}\, \mathrm{Sch}_{1+1}(B, Z \boxtimes Z') \otimes \mathrm{Sch}_{1+1}(Z \boxtimes Z', B').$$

Remark 1.2. We have $\mathrm{Irr}(\mathrm{Sch}_{(1)}) = \{Z_1\boxtimes...\boxtimes Z_k | k \geq 0, Z_1, ..., Z_k \in \mathrm{Irr(Sch)}\}$; and $\mathrm{Irr}(\mathrm{Sch}_{(2)}) = \{(Z_1\boxtimes...\boxtimes Z_k)\underline{\boxtimes}(W_1\boxtimes...\boxtimes W_\ell) | k, \ell \geq 0, Z_1, ..., W_\ell \in \mathrm{Irr(Sch)}\}$. Then $c(Z_1\boxtimes...\boxtimes Z_k) = Z_1\otimes...\otimes Z_k$ and $c((Z_1 \boxtimes ... \boxtimes Z_k)\underline{\boxtimes}(W_1 \boxtimes ... \boxtimes W_l)) = (Z_1 \otimes ... \otimes Z_k) \boxtimes (W_1 \otimes ... \otimes W_l)$.

1.2. **Quasi-categories.** We define a quasi-category $\mathcal{C}$ to be the data of: (a) a set of objects $\mathrm{Ob}(\mathcal{C})$; (b) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set of morphisms $\mathcal{C}(X, Y)$, and for any $X \in \mathrm{Ob}(\mathcal{C})$, an element $\mathrm{id}_X \in \mathcal{C}(X, X)$; (c) for $X_i \in \mathrm{Ob}(\mathcal{C})$ $(i = 1, 2, 3)$, a subset $\mathcal{C}(X_1, X_2, X_3) \subset \mathcal{C}(X_1, X_2) \times \mathcal{C}(X_2, X_3)$ and a map $\mathcal{C}(X_1, X_2, X_3) \xrightarrow{\circ} \mathcal{C}(X_1, X_3)$, $(x_1, x_2) \mapsto x_2 \circ x_1$, such that:

(identity axiom) if $X, Y \in \mathrm{Ob}(\mathcal{C})$, and $x \in \mathcal{C}(X, Y)$, then $\mathrm{id}_Y \circ x \in \mathcal{C}(X, Y, Y)$, $x \circ \mathrm{id}_X \in \mathcal{C}(X, X, Y)$, and $\mathrm{id}_Y \circ x = x \circ \mathrm{id}_X = x$;

(associativity axiom) if $X_i \in \mathrm{Ob}(\mathcal{C})$ $(i = 1, ..., 4)$ and $x_i \in \mathcal{C}(X_i, X_{i+1})$ $(i = 1, 2, 3)$, then if: $(x_1, x_2) \in \mathcal{C}(X_1, X_2, X_3)$, $(x_2 \circ x_1, x_3) \in \mathcal{C}(X_1, X_3, X_4)$, $(x_2, x_3) \in \mathcal{C}(X_2, X_3, X_4)$ and $(x_1, x_3 \circ x_2) \in \mathcal{C}(X_1, X_2, X_4)$, then $x_3 \circ (x_2 \circ x_1) = (x_3 \circ x_2) \circ x_1$.

We then define inductively a diagram $\mathcal{C}(X_1, X_2) \times ... \times \mathcal{C}(X_{n-1}, X_n) \supset \mathcal{C}(X_1, ..., X_n) \xrightarrow{\circ} \mathcal{C}(X_1, X_n)$, as follows: $(x_1, ..., x_{n-1}) \in \mathcal{C}(X_1, ..., X_n)$ iff for any $k = 2, ..., n-1$, $(x_1, ..., x_{k-1}) \in \mathcal{C}(X_1, ..., X_k)$, $(x_k, ..., x_{n-1}) \in \mathcal{C}(X_k, ..., X_n)$, and $(x_{k-1} \circ ... \circ x_1, x_{n-1} \circ ... \circ x_k) \in \mathcal{C}(X_1, X_k, X_n)$; if $(x_1, ..., x_{n-1})$ satisfies these conditions, then the $(x_{n-1} \circ ... \circ x_k) \circ (x_{k-1} \circ ... \circ x_1)$ all coincide; this defines the map $\mathcal{C}(X_1, ..., X_n) \to \mathcal{C}(X_1, X_n)$.

If $1 < n_1 < ... < n_k < n$ and $x = (x_1, ..., x_{n-1}) \in \mathcal{C}(X_1, X_2) \times ... \times \mathcal{C}(X_{n-1}, X_n)$, then $x \in \mathcal{C}(X_1, ..., X_n)$ iff: (a) $(x_1, ..., x_{n_1-1}) \in \mathcal{C}(X_1, ..., X_{n_1})$, $(x_{n_1}, ..., x_{n_2-1}) \in \mathcal{C}(X_{n_1}, ..., X_{n_2})$, ..., and $(x_{n_{k-1}}, ..., x_{n-1}) \in \mathcal{C}(X_{n_{k-1}}, ..., X_n)$; (b) moreover, $(x_{n_1-1} \circ ... \circ x_1, ..., x_{n-1} \circ ... \circ x_{n_{k-1}}) \in \mathcal{C}(X_1, X_{n_1}, X_{n_2}, ..., X_n)$. If these conditions are satisfied, then $x_{n-1} \circ ... \circ x_1 = (x_{n-1} \circ ... \circ x_{n_{k-1}+1}) \circ ... \circ (x_{n_1-1} \circ ... \circ x_1)$.

The quasi-category $\mathcal{C}$ is called strict monoidal if it is equipped with: (a) a map $\otimes : \mathrm{Ob}(\mathcal{C})^2 \to \mathcal{C}$, $(X, Y) \mapsto X \otimes Y$ and an object $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$, such that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$; (b) a map $\otimes : \mathcal{C}(X, Y) \times \mathcal{C}(X', Y') \to \mathcal{C}(X \otimes X', Y \otimes Y')$ such that $(f\otimes f')\otimes f'' = f\otimes(f'\otimes f'')$, $f\otimes\mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f$; (c) a map $\otimes : \mathcal{C}(X_1, X_2, X_3)\times\mathcal{C}(X_1', X_2', X_3') \to$

$\mathcal{C}(X_1 \otimes X_1', X_2 \otimes X_3', X_3 \otimes X_3')$, such that

$$\begin{array}{ccc} \mathcal{C}(X_1, X_2, X_3) \times \mathcal{C}(X_1', X_3', X_3') & \to & \mathcal{C}(X_1 \otimes X_1', X_2 \otimes X_2', X_3 \otimes X_3') \\ \downarrow & & \downarrow \\ \mathcal{C}(X_1, X_2) \times \mathcal{C}(X_2, X_3) \times \mathcal{C}(X_1', X_2') \times \mathcal{C}(X_2', X_3') & \to & \mathcal{C}(X_1 \otimes X_1', X_2 \otimes X_2') \times \mathcal{C}(X_2 \otimes X_2', X_3 \otimes X_3') \end{array}$$

commutes. Then we have maps $\mathcal{C}(X_1, ..., X_n) \otimes \mathcal{C}(X_1', ..., X_n') \to \mathcal{C}(X_1 \otimes X_1', ..., X_n \otimes X_n')$, such that the analogous diagram (with 3 replaced by n) commutes.

Example. $\mathcal{G}$ is the category where objects are pairs (I, J) of finite sets and $\mathcal{G}((I, J), (I', J'))$ is the set of oriented acyclic graphs with vertices $i_{in}, j_{in}, i'_{out}, j'_{out}$, $i \in I$, $j \in J$, $i' \in I'$, $j' \in J'$, where each edge has its origin in $\{i_{in}, j'_{out} | i \in I, j' \in J'\}$ and its end in $\{i'_{out}, j_{in} | i' \in I', j \in J\}$, and there is at most one edge through two given vertices. Equivalently, a graph is a subset of $(I \sqcup J') \times (I' \sqcup J)$. If $X_\alpha = (I_\alpha, J_\alpha)$ and $x_\alpha \in \mathcal{G}(X_\alpha, X_{\alpha+1})$ $(\alpha = 1, ..., k-1)$, we obtain a composed graph with edges x_{in}, y_{out}, $x \in I_1 \sqcup J_1$, $y \in I_n \sqcup J_n$, by declaring that two edges are connected if there exists an oriented path in the juxtaposition of $x_1, ..., x_{k-1}$ relating them. Then $\mathcal{G}(X_1, ..., X_k) \subset \mathcal{G}(X_1, X_2) \times ... \times \mathcal{G}(X_{k-1}, X_k)$ is the set of tuples of graphs whose composed graph is acyclic, which is then their composition. The tensor product is given by $(I, J) \otimes (I', J') := (I \sqcup J, I' \sqcup J')$ at the level of objects, and by the disjoint union of graphs at the level of morphisms. Note that $\mathcal{G}$ contains subcategories $\mathcal{G}^{\text{left}}$ and $\mathcal{G}^{\text{right}}$, where $\mathcal{G}^{\text{left}}((I, J), (I', J')) = \{S \in \mathcal{G}((I, J), (I', J')) | S \cap (J' \times I') = \emptyset\}$, and $\mathcal{G}^{\text{right}}((I, J), (I', J')) = \{S | S \cap (I \times J) = \emptyset\}$. □

A $\mathbf{k}$-additive quasi-category $\mathcal{C}$ is the data of: (a) a set of objects $\text{Ob}(\mathcal{C})$, (b) for any $X, Y \in \text{Ob}(\mathcal{C})$, a vector space $\mathcal{C}(X, Y)$, and for any $X_1, ..., X_n \in \text{Ob}(\mathcal{C})$, a vector subspace $\mathcal{C}(X_1, ..., X_n) \subset \mathcal{C}(X_1, X_2) \otimes ... \otimes \mathcal{C}(X_{n-1}, X_n)$, and a linear map $\mathcal{C}(X_1, ..., X_n) \to \mathcal{C}(X_1, X_n)$, satisfying the axioms of a quasi-category (with products replaced by tensor products); (c) an associative direct sum map $\oplus : \text{Ob}(\mathcal{C})^2 \to \text{Ob}(\mathcal{C})$, $(X, Y) \mapsto X \oplus Y$, an object $\mathbf{0} \in \text{Ob}(\mathcal{C})$, and isomorphisms $\mathcal{C}(Z, X \oplus Y) \simeq \mathcal{C}(Z, X) \oplus \mathcal{C}(Z, Y)$ and $\mathcal{C}(X \oplus Y, Z) \simeq \mathcal{C}(X, Z) \oplus \mathcal{C}(Y, Z)$, such that:

$\mathcal{C}(X_1 \oplus X_1', X_2, X_3) \simeq \mathcal{C}(X_1, X_2, X_3) \oplus \mathcal{C}(X_1', X_2, X_3)$, $\mathcal{C}(X_1, X_2, X_3 \oplus X_3') \simeq \mathcal{C}(X_1, X_2, X_3) \oplus \mathcal{C}(X_1, X_2, X_3')$, $\mathcal{C}(X_1, X_2 \oplus X_2', X_3') \simeq \mathcal{C}(X_1, X_2, X_3) \oplus \mathcal{C}(X_1, X_2', X_3) \oplus \mathcal{C}(X_1, X_2) \otimes \mathcal{C}(X_2, X_3) \oplus \mathcal{C}(X_1, X_2') \otimes \mathcal{C}(X_2, X_3)$, and the composition map on left sides coincides with the sum of compositions on the right sides, and of the zero maps on the two last summands in the last case (this statement then generalizes to $\mathcal{C}(X_1, ..., X_i \oplus X_i', ..., X_n)$);

$X \oplus \mathbf{0} = X = \mathbf{0} \oplus X$ and $\mathcal{C}(X, \mathbf{0}) = \mathcal{C}(\mathbf{0}, X) = 0$ for any X, and the composed isomorphisms $\mathcal{C}(X, Y) = \mathcal{C}(X \oplus \mathbf{0}, Y) \simeq \mathcal{C}(X, Y)$, $\mathcal{C}(X, Y) = \mathcal{C}(\mathbf{0} \oplus X, Y) \simeq \mathcal{C}(X, Y)$, $\mathcal{C}(X, Y) = \mathcal{C}(X, Y \oplus \mathbf{0}) \simeq \mathcal{C}(X, Y)$ and $\mathcal{C}(X, Y) = \mathcal{C}(X, \mathbf{0} \oplus Y) \simeq \mathcal{C}(X, Y)$ are the identity.

Such a $\mathcal{C}$ is called strict monoidal if it satisfies the above axioms of a strict monoidal quasi-category, where $\otimes$ is bilinear and biadditive.

A functor $F : \mathcal{C} \to \mathcal{D}$ between quasi-categories is defined as the data of a map $F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$, and a collection of maps $F(X, Y) : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$, such that $\times_{i=1}^{n-1} F(X_i, X_{i+1})$ restricts to a map $\mathcal{C}(X_1, ..., X_n) \to \mathcal{D}(F(X_1), ..., F(X_n))$ and the natural diagrams commute; natural additional axioms are imposed if the categories are strict monoidal and/or additive.

Example. $\mathbf{k}\mathcal{G}$ is the category with $\text{Ob}(\mathbf{k}\mathcal{G}) = \text{Ob}(\mathcal{G})$ and $(\mathbf{k}\mathcal{G})((I, J), (I', J')) = \mathbf{k}\mathcal{G}((I, J), (I', J'))$; then $\mathbf{k}\mathcal{G}$ is an additive strict monoidal quasi-category. □

1.3. **Partial traces and quasi-categories.** If $\mathcal{C}_0$ is a symmetric strict monoidal category with symmetry constraint $\beta_{X,Y} \in \mathcal{C}_0(X \otimes Y, Y \otimes X)$, a partial trace on $\mathcal{C}_0$ is the data of diagrams $\mathcal{C}_0(X \otimes Z, Y \otimes Z) \supset \mathcal{C}_0(X, Y|Z) \stackrel{\text{tr}_Z}{\to} \mathcal{C}_0(X, Y)$ for $X, Y, Z \in \text{Ob}(\mathcal{C})$, such that: $\mathcal{C}_0(X, Y|Z \otimes Z') \subset \mathcal{C}_0(X \otimes Z, Y \otimes Z|Z') \cap \text{tr}_{Z'}^{-1}(\mathcal{C}_0(X, Y|Z))$, and $\text{tr}_{Z \otimes Z'} = \text{tr}_Z \circ \text{tr}_{Z'}$; the map $x \mapsto x' :=$

$(\mathrm{id}_Y \otimes \beta_{Z',Z}) \circ x \circ (\mathrm{id}_X \otimes \beta_{Z,Z'})$ induces an isomorphism $\mathcal{C}_0(X,Y|Z\otimes Z') \to \mathcal{C}_0(X,Y|Z'\otimes Z)$, and $\mathrm{tr}_{Z'\otimes Z}(x') = \mathrm{tr}_{Z\otimes Z'}(x)$; the composition takes $\mathcal{C}_0(X,Y|T)\times\mathcal{C}_0(Y,Z)$ to $\mathcal{C}_0(X,Z|T)$, and $\mathrm{tr}_T((y\otimes \mathrm{id}_T)\circ x) = y\circ \mathrm{tr}_T(x)$, and similarly it takes $\mathcal{C}_0(X,Y)\times\mathcal{C}_0(Y,Z|T)$ to $\mathcal{C}_0(X,Z|T)$, and $\mathrm{tr}_T(y\circ(x\otimes\mathrm{id}_T)) = \mathrm{tr}_T(y)\circ x$; the map $\times_{i=1}^2 \mathcal{C}_0(X_i\otimes T_i, Y_i\otimes T_i) \to \mathcal{C}_0(X_1\otimes X_2\otimes T_1\otimes T_2, Y_1\otimes Y_2\otimes T_1\otimes T_2)$, $(x_1,x_2)\mapsto x := (\mathrm{id}_{Y_1}\otimes\beta_{T_1,Y_2}\otimes\mathrm{id}_{T_2})\circ(x_1\otimes x_2)\circ(\mathrm{id}_{X_1}\otimes\beta_{X_2,T_1}\otimes\mathrm{id}_{T_2})$ takes $\mathcal{C}_0(X_1,Y_1|T_1)\times\mathcal{C}_0(X_2,Y_2|T_2)$ to $\mathcal{C}_0(X_1\otimes X_2, Y_1\otimes Y_2|T_1\otimes T_2)$, and $\mathrm{tr}_{T_1\otimes T_2}(x) = \mathrm{tr}_{T_1}(x_1)\otimes \mathrm{tr}_{T_2}(x_2)$; $\mathcal{C}_0(X,Y|\mathbf{1}) = \mathcal{C}_0(X,Y)$ and $\mathrm{tr}_{\mathbf{1}}(x) = x$.

Set $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(\mathcal{C}_0)^2$, $(X,Y)\otimes(X',Y') := (X\otimes X', Y\otimes Y')$, $\mathcal{C}((X,Y),(X',Y')) := \mathcal{C}_0(X\otimes Y', X'\otimes Y)$, $\mathcal{C}((X_1,Y_1),(X_2,Y_2),(X_3,Y_3)) := \{(x_1,x_2)|x_2 * x_1 \in \mathcal{C}_0(X_1\otimes Y_3, X_3\otimes Y_1|Y_2)\}$, where $x_2 * x_1 = (\mathrm{id}_{X_3}\otimes\beta_{Y_2Y_1})\circ(x_2\otimes\mathrm{id}_{Y_1})\circ(\mathrm{id}_{X_2}\otimes\beta_{Y_1Y_3})\circ(x_1\otimes\mathrm{id}_{Y_3})\circ(\mathrm{id}_{X_1}\otimes\beta_{Y_3Y_2})$; then $x_2\circ x_1 := \mathrm{tr}_{Y_2}(x_2 * x_1)$. The tensor product of morphisms is defined as $\times_{i=1}^2\mathcal{C}_0(X_i\otimes Y_i', X_i'\otimes Y_i) \ni (x_1,x_2)\mapsto (\mathrm{id}_{X_1'}\otimes\beta_{Y_1,X_2'}\otimes\mathrm{id}_{Y_2})\circ(x_1\otimes x_2)\circ(\mathrm{id}_{X_1}\otimes\beta_{X_2,Y_1'}\otimes\mathrm{id}_{Y_2'}) \in \mathcal{C}_0(X_1\otimes X_2\otimes Y_1'\otimes Y_2', X_1'\otimes X_2'\otimes Y_1\otimes Y_2)$; the unit of $\mathcal{C}$ is $(\mathbf{1},\mathbf{1})$.

Proposition 1.3. *$\mathcal{C}$ is a strict monoidal quasi-category.*

Proof. Let $U_i = (X_i,Y_i)$; let $x_i\in\mathcal{C}(U_i,U_{i+1})$ $(i=1,2,3)$; assume that $(x_1,x_2)\in\mathcal{C}(U_1,U_2,U_3)$ and $(x_2\circ x_1, x_3)\in\mathcal{C}(U_1,U_3,U_4)$; define $x_3 * x_2 * x_1$ by formula (8) below. Let us show that $x_3*x_2*x_1\in\mathcal{C}(X_1\otimes Y_4, X_4\otimes Y_1|Y_2\otimes Y_3)$ and that $x_3\circ(x_2\circ x_1) = \mathrm{tr}_{Y_2\otimes Y_3}(x_3*x_2*x_1)$.

$x_3\circ(x_2\circ x_1) = \mathrm{tr}_{Y_3}(x_3 * \mathrm{tr}_{Y_2}(x_2*x_1))$. Using the fact that x_2*x_1 may as well be expressed as $x_2*x_1 = (\mathrm{id}_{X_3}\otimes\beta_{Y_2Y_1})\circ(x_2\otimes\mathrm{id}_{Y_1})\circ(\beta_{Y_3X_2}\otimes\mathrm{id}_{Y_1})\circ(\mathrm{id}_{Y_3}\otimes x_1)\circ(\beta_{X_1Y_3}\otimes\mathrm{id}_{Y_2})$, we write $x_3*\mathrm{tr}_{Y_2}(x_2*x_1) = (\mathrm{id}_{X_4}\otimes\beta_{Y_3,Y_1})\circ(x_3\otimes\mathrm{id}_{Y_1})\circ(\beta_{Y_4,X_3}\otimes\mathrm{id}_{Y_1})\circ(\mathrm{id}_{Y_4}\otimes\mathrm{tr}_{Y_2}(x_2*x_1))\circ(\beta_{X_1,Y_4}\otimes\mathrm{id}_{Y_3})$. Now $\mathrm{id}_{Y_4}\otimes(x_2*x_1)\in\mathcal{C}_0(Y_4\otimes X_1\otimes Y_3, Y_4\otimes X_3\otimes Y_1|Y_2)$, and $\mathrm{id}_{Y_4}\otimes\mathrm{tr}_{Y_2}(x_2*x_1) = \mathrm{tr}_{Y_2}(\mathrm{id}_{Y_4}\otimes(x_2*x_1))$. We have then $[\{(\mathrm{id}_{X_4}\otimes\beta_{Y_3,Y_1})\circ(x_3\otimes\mathrm{id}_{Y_1})\circ(\beta_{Y_4,X_3}\otimes\mathrm{id}_{Y_1})\}\otimes\mathrm{id}_{Y_2}]\circ[\mathrm{id}_{Y_4}\otimes(x_2*x_1)]\circ[\beta_{X_1,Y_4}\otimes\mathrm{id}_{Y_3}\otimes\mathrm{id}_{Y_2}]\in\mathcal{C}_0(X_1\otimes Y_4\otimes Y_3, X_4\otimes Y_1\otimes Y_3|Y_2)$, and $x_3*\mathrm{tr}_{Y_2}(x_2*x_1) = \mathrm{tr}_{Y_2}\{[\{(\mathrm{id}_{X_4}\otimes\beta_{Y_3,Y_1})\circ(x_3\otimes\mathrm{id}_{Y_1})\circ(\beta_{Y_4,X_3}\otimes\mathrm{id}_{Y_1})\}\otimes\mathrm{id}_{Y_2}]\circ[\mathrm{id}_{Y_4}\otimes(x_2*x_1)]\circ[\beta_{X_1,Y_4}\otimes\mathrm{id}_{Y_3}\otimes\mathrm{id}_{Y_2}]\}$. As the right side is in the domain of tr_{Y_3}, the argument of tr_{Y_2} in the right side is in the domain of $\mathrm{tr}_{Y_3\otimes Y_2}$, and $x_3\circ(x_2\circ x_1) = \mathrm{tr}_{Y_3\otimes Y_2}\{[\{(\mathrm{id}_{X_4}\otimes\beta_{Y_3,Y_1})\circ(x_3\otimes\mathrm{id}_{Y_1})\circ(\beta_{Y_4,X_3}\otimes\mathrm{id}_{Y_1})\}\otimes\mathrm{id}_{Y_2}]\circ[\mathrm{id}_{Y_4}\otimes(x_2*x_1)]\circ[\beta_{X_1,Y_4}\otimes\mathrm{id}_{Y_3}\otimes\mathrm{id}_{Y_2}]\}$. On the other hand, this argument also expressed as $(\mathrm{id}_{X_1}\otimes\mathrm{id}_{Y_4}\otimes\beta_{Y_2,Y_3})\circ(x_3*x_2*x_1)\circ(\mathrm{id}_{X_1}\otimes\mathrm{id}_{Y_4}\otimes\beta_{Y_3,Y_2})$, therefore $x_3*x_2*x_1$ is in the domain of $\mathrm{tr}_{Y_2\otimes Y_3}$ and $x_3\circ(x_2\circ x_1) = \mathrm{tr}_{Y_2\otimes Y_3}(x_3*x_2*x_1)$. One proves in the same way that $(x_3\circ x_2)\circ x_1 = \mathrm{tr}_{Y_2\otimes Y_3}(x_3*x_2*x_1)$, which proves the asociativity identity. ☐

More generally, one shows that for any $(x_1,...,x_{n-1})\in\mathcal{C}((X_1,Y_1),...,(X_n,Y_n))$, we have $x_{n-1}*...*x_1\in\mathcal{C}_0(X_1\otimes Y_n, X_n\otimes Y_1|Y_2\otimes...\otimes Y_{n-1})$, where $x_{n-1}*...*x_1\in\mathcal{C}_0(X_1\otimes Y_n\otimes Y_2\otimes...\otimes Y_{n-1}, X_n\otimes Y_1\otimes Y_2\otimes...\otimes Y_{n-1})$ is defined inductively by

$$\begin{aligned} x_n*...*x_1 &:= (\mathrm{id}_{X_{n+1}}\otimes\beta_{Y_n,Y_1\otimes...\otimes Y_{n-1}})\circ(x_n\otimes\mathrm{id}_{Y_1\otimes...\otimes Y_{n-1}})\circ(\mathrm{id}_{X_n}\otimes\beta_{Y_1\otimes...\otimes Y_{n-1},Y_{n+1}}) \\ &\circ[(x_{n-1}*...*x_1)\otimes\mathrm{id}_{Y_{n+1}}]\circ(\mathrm{id}_{X_1}\circ\beta_{Y_{n+1},Y_2\otimes...\otimes Y_{n-1},Y_n}), \end{aligned} \tag{8}$$

where $\beta_{X,Y,Z}\in\mathcal{C}_0(X\otimes Y\otimes Z, Z\otimes Y\otimes X)$ is $\beta_{X\otimes Y,Z}\circ(\beta_{X,Y}\otimes\mathrm{id}_Z)$, and that $x_{n-1}\circ...\circ x_1 := \mathrm{tr}_{Y_2\otimes...\otimes Y_{n-1}}(x_{n-1}*...*x_1)$.

If $X\mapsto X^*$ is an involution of $\mathcal{C}_0$, another symmetric strict monoidal quasi-category $\mathcal{C}'$ may be defined by $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C}_0)^2$, $\mathcal{C}'((X,Y),(X',Y')) := \mathcal{C}_0(X\otimes Y'^*, X'\otimes Y^*)$.

A functor between categories with partial traces is a tensor functor $F:\mathcal{C}_0\to\mathcal{D}_0$, such that $F(\mathcal{C}_0(X,Y|Z))\subset\mathcal{D}_0(F(X),F(Y)|F(Z))$, and such that $\mathrm{tr}_{F(Z)}\circ F = F\circ\mathrm{tr}_Z$ (equality of maps $\mathcal{C}_0(X,Y|Z)\to\mathcal{D}_0(F(X),F(Y)))$. Such a functor induces a functor $\mathcal{C}\to\mathcal{D}$ between the corresponding quasi-categories.

If now $\mathcal{C}_0$ is additive and is a free module category over Vect, this construction can be extended as follows. In the definition of a trace, the maps are now linear and products are replaced by tensor products. Let $\mathrm{Ob}'(\mathcal{C}_0)\subset\mathrm{Ob}(\mathcal{C}_0)$ be a set of generators, i.e., each $F\in\mathrm{Ob}(\mathcal{C}_0)$ has the form $\oplus_{X\in\mathrm{Ob}'(\mathcal{C}_0)}F_X\otimes X$, where $X\mapsto F_X$ is a finitely supported map $\mathrm{Ob}'(\mathcal{C}_0)\to$

Ob(Vect). Then $\mathrm{Ob}(\mathcal{C}) := \{$finitely supported maps $\mathrm{Ob}'(\mathcal{C}_0)^2 \to \mathrm{Vect}$, $F = [(X,Y) \mapsto F_{X,Y}]\}$. We then set $\mathcal{C}(F,G) := \oplus_{(X,Y),(X',Y')} \mathrm{Vect}(F_{X,Y}, G_{X',Y'}) \otimes \mathcal{C}_0(X \otimes Y', X' \otimes Y)$ and extend the above composition and tensor product operations by linearity. In the case of $\mathcal{C}'$, we replace $\mathcal{C}_0(X \otimes Y', X' \otimes Y)$ by $\mathcal{C}_0(X \otimes Y'^*, X' \otimes Y^*)$.

Example. Let $\mathcal{G}_0$ be the category where objects are finite sets, and $\mathcal{G}_0(I,J) = \{$subsets of $I \times J\}$, and composition given by $S' \circ S :=$ the image in $I \times I''$ of $S \times_{I'} S'$, for $S \subset I \times I'$ and $S' \subset I' \times I''$; to $S \subset I \times J$, we associate the oriented graph with vertices $I \sqcup J$ and egdes $i \to j$ if $(i,j) \in S$, composition then corresponds to the composition of graphs. The tensor product is $I \otimes I' := I \sqcup I'$ and for $S \in \mathcal{G}_0(I,I')$, $T \in \mathcal{G}_0(J,J')$, $S \otimes T := S \sqcup T \subset (I \times I') \sqcup (J \times J') \subset (I \sqcup I') \times (J \sqcup J')$. Then $\mathcal{G}_0$ is a strict monoidal category. It has a partial trace defined as follows. For I, J, K finite sets, let $\mathcal{G}_0(I, J|K) \subset \mathcal{G}_0(I \sqcup K, J \sqcup K)$ be the set of graphs, such that the introduction of the edges $k_{out} \to k_{in}$ $(k \in K)$ does not introduce cycles (alternatively, the set of $S \subset (I \sqcup K) \times (J \sqcup K)$, such that the relation in K defined by $u \prec v$ if $(u,v) \in S$, has no cycle), and if x is such a graph, then $\mathrm{tr}_K(x) \in \mathcal{G}_0(I,J)$ corresponds to $\{(i,j) \in I \times J|$ there exists $s \geq 0$ and a sequence $(k_1, ..., k_s)$ of elements of K, such that $i \prec k_1 \prec ... \prec k_s \prec j\}$, where the relation $\prec$ is extended to $I \sqcup K \sqcup J$ by $u \prec v$ iff $(u,v) \in S$. Then the strict monoidal quasi-category constructed from $\mathcal{G}_0$, equipped with its partial trace, coincides with $\mathcal{G}$.

Here is another description of $\mathrm{tr}_K(x)$. As the relation $\prec$ on K is acyclic, we may extend it to a total order relation $<$ on K. Extend it to $I \sqcup K \sqcup J$ by $i < k < j$ for any $i, j, k \in I, J, K$. The relation $<$ induces a numbering $K = \{k_1, ..., k_{|K|}\}$ for K, where $k_1 < ... < k_{|K|}$. For $\alpha \in [|K|]$, let $K_\alpha := \{k_\alpha\} \sqcup \{(u,v) \in (I \sqcup K \sqcup J)^2 | u \prec v, u < k_\alpha < v\} \in \mathrm{Ob}(\mathcal{G}_0)$. Then $\mathrm{tr}_K(x) = x_{K_{|K|}J} \circ ... \circ x_{K_1K_2} \circ x_{IK_1}$, where:

• $x_{K_\alpha K_{\alpha+1}} \in \mathcal{G}_0(K_\alpha, K_{\alpha+1})$ is defined as follows: we have identifications $K_\alpha \simeq \{k_\alpha\} \sqcup K'_{\alpha,\alpha+1} \sqcup K_{\alpha,\alpha+1}$ and $K_{\alpha+1} \simeq \{k_{\alpha+1}\} \sqcup K''_{\alpha,\alpha+1} \sqcup K_{\alpha,\alpha+1}$, where $K'_{\alpha,\alpha+1} := \{s \in I \sqcup K | s < k_\alpha, s \prec k_{\alpha+1}\}$, $K''_{\alpha,\alpha+1} := \{t \in K \sqcup J | t > k_{\alpha+1}, t \succ k_\alpha\}$, $K_{\alpha,\alpha+1} := \{(s,t) \in (I \sqcup K) \times (K \sqcup J) | s < k_\alpha, k_{\alpha+1} < t, s \prec t\}$; let $\diamond$ be a one-element set if $k_\alpha \prec k_{\alpha+1}$ and $\emptyset$ otherwise; then we define $\kappa_{\alpha,\alpha+1} := \{k_\alpha\} \times (\diamond \sqcup K''_{\alpha+1}) \in \mathcal{G}_0(\{k_\alpha\}, \diamond \sqcup K''_{\alpha,\alpha+1})$ and $\lambda_{\alpha,\alpha+1} := (\diamond \sqcup K'_{\alpha,\alpha+1}) \times \{k_{\alpha+1}\} \in \mathcal{G}_0(\diamond \sqcup K'_{\alpha,\alpha+1}, \{k_{\alpha+1}\})$; then $x_{K_\alpha K_{\alpha+1}} := [(\lambda_{\alpha,\alpha+1} \otimes \mathrm{id}_{K''_{\alpha,\alpha+1}}) \circ (\mathrm{id}_\diamond \otimes \beta_{K''_{\alpha,\alpha+1}, K'_{\alpha,\alpha+1}}) \circ (\kappa_{\alpha,\alpha+1} \otimes \mathrm{id}_{K'_{\alpha,\alpha+1}})] \otimes \mathrm{id}_{K_{\alpha,\alpha+1}}$;

• $x_{IK_1} \in \mathcal{G}_0(I, K_1)$ is defined as follows: $K_1 \simeq \{k_1\} \sqcup (\sqcup_{i \in I} K''_i)$, where $K''_i := \{t \in K \sqcup J | t \neq k_1, t \succ i\}$; set $\diamond := \{i \in I | i \prec k_1\}$ and $\diamond_i := \diamond \cap \{i\}$ so $\diamond = \sqcup_{i \in I} \diamond_i$; let $\kappa_i := \{i\} \times (\diamond_i \sqcup K''_i) \in \mathcal{G}_0(\{i\}, \diamond_i \sqcup K''_i)$, $br \in \mathcal{G}_0(\sqcup_i (*_i \sqcup K''_i), * \sqcup (\sqcup_{i \in I} K''_i))$ be the canonical braiding morphism; and let $\lambda_{01} := \diamond \times \{k_1\} \in \mathcal{G}_0(\diamond, \{k_1\})$; then $x_{IK_1} := (\lambda_{01} \otimes (\otimes_{i \in I} \mathrm{id}_{K''_i})) \circ br \circ (\otimes_{i \in I} \kappa_i)$;

• $x_{K_{|K|}J} \in \mathcal{G}_0(K_{|K|}, J)$ is defined as follows: $K_{|K|} \simeq \{k_{|K|}\} \sqcup (\sqcup_{j \in J} K'_j)$, where $K'_j := \{s \in I \sqcup K | s \neq k_{|K|}, s \prec j\}$, set $\diamond := \{j \in J | j \succ k_{|K|}\}$; let $\diamond_J := \diamond \cap \{j\}$, then $\diamond = \sqcup_{j \in J} \diamond_j$; let $\kappa_{|K|,|K|+1} := \{k_{|K|}\} \times \diamond \in \mathcal{G}_0(\{k_{|K|}\}, \diamond)$; let $\lambda_j := K'_j \times \{j\} \in \mathcal{G}_0(K'_j, \{j\})$; let $br \in \mathcal{G}_0(\diamond \sqcup (\sqcup_{j \in J} K'_j), \sqcup_{j \in J} (\diamond_j \sqcup K'_j))$ be the canonical braiding map; then $x_{K_{|K|}J} := (\otimes_{j \in J} \lambda_j) \circ br \circ (\kappa_{|K|,|K|+1} \otimes (\otimes_{j \in J} \mathrm{id}_{K'_j}))$.

1.4. **Props and (quasi)(bi)(multi)props.** A prop P is a symmetric additive strict monoidal category, equipped with a tensor functor $i_P : \mathrm{Sch} \to P$, inducing a bijection on the sets of objects; so $\mathrm{Ob}(P) = \mathrm{Ob}(\mathrm{Sch})$ (see, e.g., [Tam]). It is easy to check that this definition is equivalent to the original one ([McL]). For $\phi \in \mathrm{Sch}(F,G) \to P(F,G)$, we sometimes write ϕ instead of $i_P(\phi)$. A prop morphism $f : P \to Q$ is a tensor functor, inducing a bijection on the sets of objects, and such that $f \circ i_P = i_Q$.

A biprop (resp., multiprop, bi-multiprop) is a symmetric additive monoidal category π (resp., Π^0, Π), equipped with a tensor functor $\mathrm{Sch}_{1+1} \to \pi$ (resp., $\mathrm{Sch}_{(1)} \to \Pi^0$, $\mathrm{Sch}_{(1+1)} \to \Pi$), which induces a bijection on the sets of objects. Morphisms betweens these structures are defined as above.

A quasi-prop is a symmetric additive strict monoidal quasi-category P, equipped with a morphism $i_P : \mathrm{Sch} \to P$, inducing a bijection on the sets of objects. Quasi(bi)(multi)props are defined in the same way, as well as morphisms between these structures.

A topological (quasi)(bi)(multi)prop is defined in the same way as its non-topological analogue, replacing Sch_* by $\mathbf{Sch}_*$. E.g., a topological prop $\mathbf{P}$ is a symmetric tensor category, equipped with a morphism $\mathbf{Sch} \to \mathbf{P}$, which is the identity on objects.

1.5. **Operations on props.** If $H \in \mathrm{Ob}(\mathrm{Sch})$ (resp., $\mathrm{Ob}(\mathrm{Sch}_{1+1})$) and P is a (bi)prop, then we define a prop $H(P)$ by $H(P)(F,G) := P(F \circ H, G \circ H)$. A (bi)prop morphism $P \to Q$ gives rise to a prop morphism $H(P) \to H(Q)$. Similarly, if $\mathbf{P}$ is a topological (bi)prop, then for any $H \in \mathrm{Ob}(\mathbf{Sch})$ (resp., $\mathrm{Ob}(\mathbf{Sch}_{1+1})$), we get a prop $H(\mathbf{P})$, such that $H(\mathbf{P})(F,G) := \mathbf{P}(F \circ H, G \circ H)$. A morphism of topological (bi)props $\mathbf{P} \to \mathbf{Q}$ then gives rise to a prop morphism $H(\mathbf{P}) \to H(\mathbf{Q})$.

To each (quasi)(bi)multiprop Π, one associates a (quasi)(bi)prop π by $\pi(F,F') := \Pi(F,F')$, i.e., using the injections $\mathrm{Ob}(\mathrm{Sch}) = \mathrm{Ob}(\mathrm{Sch}_1) \subset \mathrm{Ob}(\mathrm{Sch}_{(1)})$ in the "non-bi" case, and $\mathrm{Ob}(\mathrm{Sch}_{1+1}) \subset \mathrm{Ob}(\mathrm{Sch}_{(1+1)})$, $F \mapsto (F_{k,l})$, where $F_{k,l} = 0$ if $(k,l) \neq (1,1)$ and $F_{1,1} = F$, in the "bi" case.

If P is a prop, then one defines a multiprop Π_P^0 by $\Pi_P^0(F,G) := P(c(F), c(G))$. The tensor product is induced by the tensor product of P and the identity $c(F \boxtimes F') = c(F) \otimes c(F')$.

1.6. **Presentation of a prop.** If P is a prop, then a prop ideal I_P of P is a collection of vector subspaces $I_P(F,G) \subset P(F,G)$, such that $(F,G) \mapsto P(F,G)/I_P(F,G)$ is a prop, which we denote by P/I_P. Then $P \to P/I_P$ is a prop morphism.

If P is a prop, $(F_i, G_i)_{i \in I}$ is a collection of pairs of Schur functors and $V_i \subset P(F_i, G_i)$ are vector subspaces, then $(V_i, i \in I)$ is the smallest of all prop ideals I_P of P, such that $V_i \subset I_P(F_i, G_i) \subset P(F_i, G_i)$ for any $i \in I$.

Let $(F_i, G_i)_{i \in I}$ be a collection of Schur functors, and let $(V_i)_{i \in I}$ be a collection of vector spaces. Then there exists a unique (up to isomorphism) prop $\mathcal{F} = \mathrm{Free}(V_i, F_i, G_i, i \in I)$, which is initial in the category of all props P equipped with linear maps $V_i \to P(F_i, G_i)$. We call it the free prop generated by (V_i, F_i, G_i).

If (F'_α, G'_α) is a collection of Schur functors and $R_\alpha \subset \mathcal{F}(F'_\alpha, G'_\alpha)$ is a collection of vector spaces, then the prop with generators (V_i, F_i, G_i) with relations $(R_\alpha, F'_\alpha, G'_\alpha)$ is the quotient of $\mathcal{F}$ with the prop ideal generated by R_α.

1.7. **Topological props.** Let P be a prop equipped with a filtration $P(F,G) = P^0(F,G) \supset P^1(F,G) \supset \ldots$ for any $F, G \in \mathrm{Ob}(\mathrm{Sch})$, compatible with direct sums and such that:

(a) $\circ$ and $\otimes$ induce maps $\circ : P^i(F,G) \otimes P^j(G,H) \to P^{i+j}(F,H)$, and $\otimes : P^i(F,G) \otimes P^{i'}(F',G') \to P^{i+i'}(F \otimes F', G \otimes G')$

(b) if $F, G \in \mathrm{Ob}(\mathrm{Sch})$ are homogeneous, then $P(F,G) = P^{||F|-|G||}(F,G)$.

For $F, G \in \mathrm{Ob}(\mathrm{Sch})$, we then define $\widehat{P}(F,G) = \lim_{\leftarrow} P(F,G)/P^n(F,G)$ as the completed separated of $P(F,G)$ w.r.t. the filtration $P^n(F,G)$. Then $\widehat{P}$ is a prop.

If $F, G \in \mathrm{Ob}(\mathbf{Sch})$, define $\mathbf{P}(F,G)$ as follows: for $F = \widehat{\oplus}_{i \geq 0} F_i$, $G = \widehat{\oplus}_{i \geq 0} G_i$ the decompositions of F, G into sums of homogeneous components, we set $\mathbf{P}(F,G) = \widehat{\oplus}_{i,j \geq 0} \widehat{P}(F_i, G_j)$ (where $\widehat{\oplus}$ is the direct product).

Proposition 1.4. $\mathbf{P}$ *is a symmetric additive strict monoidal category, equipped with a morphism* $\mathbf{Sch} \to \mathbf{P}$*, which is the identity on objects.*

Recall that $\mathbf{P}$ is called a topological prop.

Proof. Let $F = \widehat{\oplus}_i F_i$, $G = \widehat{\oplus}_i G_i$, $H = \oplus_i H_i$ be in $\mathbf{Sch}$. We define a map $\circ : \mathbf{P}(F,G) \otimes \mathbf{P}(G,H) \to \mathbf{P}(F,H)$ as follows. We first define a map $\mathbf{P}(F_i, G) \otimes \mathbf{P}(G, H_k) \to \widehat{P}(F_i, H_k)$. The left vector space injects in $\widehat{\oplus}_j \widehat{P}(F_i, G_j) \otimes \widehat{P}(G_j, H_k)$. The composition takes the jth summand to $P^{|j-i|+|j-k|}(F_i, H_k)$. As $|j-i| + |j-k| \to \infty$ as $j \to \infty$, we have a well-defined map

$\mathbf{P}(F_i, G) \otimes \mathbf{P}(G, H_k) \to \widehat{P}(F_i, H_k)$. The direct product of these maps then induces a map $\circ : \mathbf{P}(F, G) \otimes \mathbf{P}(G, H) \to \mathbf{P}(F, H)$.

Let now $F' = \oplus_i F'_i$, $G' = \oplus_i G'_i$ be in **Sch**. We define a map $\otimes : \mathbf{P}(F, G) \otimes \mathbf{P}(F', G') \to \mathbf{P}(F \otimes F', G \otimes G')$ as the direct product of the maps $\widehat{P}(F_i, G_j) \otimes \widehat{P}(F'_{i'}, G'_{j'}) \to \widehat{P}(F_i \otimes F'_{i'}, G_j \otimes G'_{j'})$. This is well-defined since $(F \otimes F')_i = \oplus_{j=0}^n F_j \otimes F'_{i-j}$ is the sum of a finite number of tensor products, and the same holds for $(G \otimes G')_j$. $\square$

A grading of P by an abelian semigroup Γ is a decomposition $P(F, G) = \oplus_{\gamma \in \Gamma} P_\gamma(F, G)$, such that the prop operations are compatible with the semigroup structure of Γ. Then if P is graded by $\mathbb{N}$ and if we set $P^n(F, G) = \oplus_{i \geq n} P_i(F, G)$, the descending filtration $P = P^0 \supset \ldots$ satisfies condition (a) above.

If $P \to Q$ is a surjective prop morphism (i.e., the maps $P(F, G) \to Q(F, G)$ are all surjective), and if P is equipped with a filtration as above, then so is Q (we define $Q^n(F, G)$ as the image of $P^n(F, G)$). Then we get a morphism $\mathbf{P} \to \mathbf{Q}$ of topological props, i.e., a morphism of tensor categories such that the morphisms $\mathbf{Sch} \to \mathbf{P} \to \mathbf{Q}$ and $\mathbf{Sch} \to \mathbf{Q}$ coincide.

If P, R are props equipped with a filtration as above, and $P \to R$ is a prop morphism compatible with the filtration (i.e., $P^n(F, G)$ maps to $R^n(F, G)$), then we get a morphism of topological props $\mathbf{P} \to \mathbf{R}$.

1.8. **Modules over props.** If $\mathcal{S}$ is an additive symmetric strict monoidal category, and $V \in \mathrm{Ob}(\mathcal{S})$, then we have a prop $\mathrm{Prop}(V)$, s.t. $\mathrm{Prop}(V)(F, G) = \mathrm{Hom}_{\mathcal{S}}(F(V), G(V))$. Then a P-module (in the category $\mathcal{S}$) is a pair (V, ρ), where $V \in \mathrm{Ob}(\mathcal{S})$ and $\rho : P \to \mathrm{Prop}(V)$ is a tensor functor. Then P-modules in the category $\mathcal{S}$ form a category. The tautological P-module is $\mathcal{S} = P$, $V = \mathbf{id}$.

1.9. **Examples of props.** We will define several props by generators and relations.

1.9.1. *The prop* Bialg. This is the prop with generators $m \in \mathrm{Bialg}(T_2, \mathbf{id})$, $\Delta \in \mathrm{Bialg}(\mathbf{id}, T_2)$, $\eta \in \mathrm{Bialg}(\mathbf{1}, \mathbf{id})$, $\varepsilon \in \mathrm{Bialg}(\mathbf{id}, \mathbf{1})$, and relations

$$m \circ (m \otimes \mathrm{id}_{\mathbf{id}}) = m \circ (\mathrm{id}_{\mathbf{id}} \otimes m), \quad (\Delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \Delta = (\mathrm{id}_{\mathbf{id}} \otimes \Delta) \circ \Delta,$$
$$\Delta \circ m = (m \otimes m) \circ (1324) \circ (\Delta \otimes \Delta),$$
$$m \circ (\eta \otimes \mathrm{id}_{\mathbf{id}}) = m \circ (\mathrm{id}_{\mathbf{id}} \otimes \eta) = \mathrm{id}_{\mathbf{id}}, \quad (\varepsilon \otimes \mathrm{id}_{\mathbf{id}}) \circ \Delta = (\mathrm{id}_{\mathbf{id}} \otimes \varepsilon) \circ \Delta = \mathrm{id}_{\mathbf{id}}.$$

When $\mathcal{S} = \mathrm{Vect}$, the category of Bialg-modules is that of bialgebras.

1.9.2. *The prop* COB. This is the prop with generators $m \in \mathrm{COB}(T_2, \mathbf{id})$, $\Delta \in \mathrm{COB}(\mathbf{id}, T_2)$, $\eta \in \mathrm{COB}(\mathbf{1}, \mathbf{id})$, $\varepsilon \in \mathrm{COB}(\mathbf{id}, \mathbf{1})$ and $R \in \mathrm{COB}(\mathbf{1}, T_2)$, and relations: $m, \Delta, \eta, \varepsilon$ satisfy the relations of Bialg,

$$(m \otimes m) \circ (1324) \circ (R \otimes ((21) \circ R)) = (m \otimes m) \circ (1324) \circ ((21) \circ R) \otimes R) = \eta \otimes \eta,$$
$$(21) \circ (m \otimes m) \circ (1324) \circ (\Delta \otimes R) = (m \otimes m) \circ (1324) \circ (R \otimes \Delta),$$
$$m^{\otimes 3} \circ (142536) \circ \big((R \otimes \eta) \otimes ((\Delta \otimes \mathrm{id}_{\mathbf{id}}) \circ R)\big) = m^{\otimes 3} \circ (142536) \circ \big((\eta \otimes R) \boxtimes (\mathrm{id}_{\mathbf{id}} \otimes \Delta) \circ R)\big)$$

The category of COB-modules over $\mathcal{S} = \mathrm{Vect}$ is that of coboundary bialgebras, i.e., pairs (A, R_A), where A is a bialgebra, and $R_A \in A^{\otimes 2}$ satisfies $R_A R_A^{21} = R_A^{21} R_A = 1_A^{\otimes 2}$, $\Delta_A^{21}(x) R_A^{21} = R_A \Delta_A(x)$, and

$$(R_A \otimes 1_A)\big((\Delta_A \otimes \mathrm{id}_A)(R_A)\big) = (1_A \otimes R_A)\big((\mathrm{id}_A \otimes \Delta_A)(R_A)\big).$$

1.9.3. *The prop* LA. This is the prop with generator the bracket $\mu \in \mathrm{LA}(\wedge^2, \mathbf{id})$ and relation the Jacobi identity

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ ((123) + (231) + (312)) = 0. \tag{9}$$

When $\mathcal{S} = \mathrm{Vect}$, the category of LA-modules is that of Lie algebras.

1.9.4. *The prop* LCA. This is the prop with generator the cobracket $\delta \in \mathrm{LCA}(\mathbf{id}, \wedge^2)$ and relation the co-Jacobi identity

$$((123)+(231)+(312)) \circ (\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \delta = 0.$$

When $\mathcal{S} = \mathrm{Vect}$, the category of LCA-modules is that of Lie coalgebras.

1.9.5. *The prop* LBA. This is the prop with generators $\mu \in \mathrm{LBA}(\wedge^2, \mathbf{id})$, $\delta \in \mathrm{LBA}(\mathbf{id}, \wedge^2)$; relations are the Jacobi and the co-Jacobi identities, and the cocycle relation

$$\delta \circ \mu = ((12)-(21)) \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (\mathrm{id}_{\mathbf{id}} \otimes \delta) \circ ((12)-(21)).$$

When $\mathcal{S} = \mathrm{Vect}$, the category of LBA-modules is that of Lie bialgebras.

1.9.6. *The prop* LBA_f. This is the prop with generators $\mu \in \mathrm{LBA}_f(\wedge^2, \mathbf{id})$, $\delta \in \mathrm{LBA}_f(\mathbf{id}, \wedge^2)$, $f \in \mathrm{LBA}_f(\mathbf{1}, \wedge^2)$ and relations: μ, δ satisfy the relations of LBA, and

$$\big((123)+(231)+(312)\big) \circ \Big((\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ f + (\mu \otimes \mathrm{id}_{\mathbf{id}^{\otimes 2}}) \circ (1324) \circ (f \otimes f)\Big) = 0.$$

The category of LBA_f-modules is the category of pairs $(\mathfrak{a}, f_{\mathfrak{a}})$ where $\mathfrak{a}$ is a Lie bialgebra and $f_{\mathfrak{a}}$ is a twist of $\mathfrak{a}$.

1.9.7. *The prop* Cob. This is the prop with generators $\mu \in \mathrm{Cob}(\wedge^2, \mathbf{id})$ and $\rho \in \mathrm{Cob}(\mathbf{1}, \wedge^2)$ and relations: μ satisfies the Jacobi identity (9), and the element $Z \in \mathrm{Cob}(\mathbf{1}, \wedge^3)$ defined by

$$Z := ((123)+(231)+(312)) \circ (\mathrm{id}_{\mathbf{id}} \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (\rho \otimes \rho)$$

is invariant, i.e., it satisfies

$$\Big((\mu \otimes \mathrm{id}_{\mathbf{id}^{\otimes 2}}) \circ (1423) + (\mathrm{id}_{\mathbf{id}} \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (1243) + (\mathrm{id}_{\mathbf{id}^{\otimes 2}} \otimes \mu)\Big) \circ (Z \otimes \mathrm{id}_{\mathbf{id}}) = 0.$$

The category of Cob-modules over $\mathcal{S} = \mathrm{Vect}$ is that of coboundary Lie bialgebras, i.e., pairs $(\mathfrak{a}, \rho_{\mathfrak{a}})$, where $\mathfrak{a}$ is a Lie algebra and $\rho_{\mathfrak{a}} \in \wedge^2(\mathfrak{a})$ is such that $Z_{\mathfrak{a}} := [\rho_{\mathfrak{a}}^{12}, \rho_{\mathfrak{a}}^{13}] + [\rho_{\mathfrak{a}}^{12}, \rho_{\mathfrak{a}}^{23}] + [\rho_{\mathfrak{a}}^{13}, \rho_{\mathfrak{a}}^{23}]$ is $\mathfrak{a}$-invariant.

1.10. **Some prop morphisms.** We have unique prop morphisms Cob $\to$ Sch, LBA $\to$ Sch and $\mathrm{LBA}_f \to$ Sch, respectively defined by $(\mu, \rho) \mapsto (0,0)$, $(\mu, \delta) \mapsto (0,0)$ and $(\mu, r) \mapsto (0,0)$.

If $\mathrm{LA} \to P$ is a prop morphism, and $\alpha \in P(\mathbf{1}, T_2)$, define

$$\mathrm{ad}(\alpha) := \Big(((\mu \circ \mathrm{Alt}) \otimes \mathrm{id}_{\mathbf{id}}) \circ (132) + \mathrm{id}_{\mathbf{id}} \otimes (\mu \circ \mathrm{Alt})\Big) \circ (\alpha \otimes \mathrm{id}_{\mathbf{id}});$$

(here $\mathrm{Alt} : T_2 \to \wedge^2$ is the alternation map); this is a propic version of the map $x \mapsto [\alpha_{\mathfrak{a}}, x^1 + x^2]$, where $\mathfrak{a} \in \mathrm{Rep}(P)$. If $\alpha \in P(\mathbf{1}, \wedge^2)$, then $\mathrm{ad}(\alpha) \in P(\mathbf{1}, \wedge^2)$.

Using the presentations of LBA and LBA_f, we get:

Proposition 1.5. *We have unique prop morphisms*

$$\kappa_1, \kappa_2 : \mathrm{LBA} \to \mathrm{LBA}_f, \quad \textit{such that} \quad \kappa_1(\mu) = \kappa_2(\mu) = \mu,\ \kappa_1(\delta) = \delta,\ \kappa_2(\delta) = \delta + \mathrm{ad}(f)$$

and a unique prop morphism

$$\kappa_0 : \mathrm{LBA}_f \to \mathrm{LBA}, \quad \textit{such that} \quad \kappa_0(\mu) = \mu,\ \kappa_0(\delta) = \delta,\ \kappa_0(f) = 0.$$

We also have a prop morphism

$$\kappa : \mathrm{LBA}_f \to \mathrm{Cob},$$

such that $\mu \mapsto \mu$, $\delta \mapsto \mathrm{ad}(\rho)$, $f \mapsto -2\rho$ and $\tau_{\mathrm{LBA}} : \mathrm{LBA} \to \mathrm{LBA}$, defined by $(\mu, \delta) \mapsto (\mu, -\delta)$.

1.10.1. *The prop* Sch. Sch it itself a prop (with no generator and relation). The corresponding category of modules over $\mathcal{S}$ is $\mathcal{S}$ itself.

1.11. **Examples of topological props.**

1.11.1. *The prop* **Sch**. We set $\mathrm{Sch}^0(F,G) = \mathrm{Sch}(F,G)$, $\mathrm{Sch}^1(F,G) = ... = 0$. This filtration satisfies conditions (a) and (b) of Section 1.7, since for F,G homogeneous, $\mathrm{Sch}(F,G) = 0$ unless F and G have the same degree. The corresponding completion of Sch coincides with **Sch**.

1.11.2. *The props* **LA** *and* **LCA**. Since the relation in LA is homogeneous in μ, the prop LA has a grading $\deg_\mu$. If $F,G \in \mathrm{Ob}(\mathrm{Sch})$ and $x \in \mathrm{LA}(F,G)$ are homogeneous, then $|G| - |F| = -\deg_\mu(x)$, which implies that the filtration induced by $\deg_\mu$ satisfies conditions (a) and (b) above. We denote by **LA** the corresponding topological prop.

In the same way, LCA has a grading $\deg_\delta$, $|G| - |F| = \deg_\delta(x)$, so the filtration induced by $\deg_\delta$ satisfies conditions (a) and (b) above. We denote by **LCA** the corresponding topological prop.

1.11.3. *The prop* **LBA**. Since the relations in LBA are homogeneous in both μ and δ, the prop LBA is equipped with a grading $(\deg_\mu, \deg_\delta)$ by $\mathbb{N}^2$. Moreover, if $F,G \in \mathrm{Ob}(\mathrm{Sch})$ and $x \in \mathrm{LBA}(F,G)$ are homogeneous, then

$$|G| - |F| = \deg_\delta(x) - \deg_\mu(x). \tag{10}$$

Then $\deg_\mu + \deg_\delta$ is a grading of LBA by $\mathbb{N}$. The corresponding filtration therefore satisfies condition (a) above. (10) also implies that it satisfies condition (b), since $\deg_\mu$ and $\deg_\delta$ are ≥ 0. We denote by **LBA** the resulting topological prop.

Let $\mathcal{LBA}$ be the category of Lie bialgebras over $\mathbf{k}$. Let $\mathcal{S}_1$ be the category of topological $\mathbf{k}[[\hbar]]$-modules (i.e., quotients of modules of the form $V[[\hbar]]$, where $V \in \mathrm{Vect}$ and the topology is given by the images of $\hbar^n V[[\hbar]]$) and let $\mathcal{S}_2$ be the category of modules of the same form, where V is a complete separated $\mathbf{k}$-vector space.

Then we have a functor $\mathcal{LBA} \to \{S(\mathbf{LBA})\text{-modules over } \mathcal{S}_1\}$, taking $\mathfrak{a}$ to $S(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{LBA})$ is given by $\mu \mapsto \mu_\mathfrak{a}$, $\delta \mapsto \hbar\delta_\mathfrak{a}$.

We also have a functor $\mathcal{LBA} \to \{S(\mathbf{LBA})\text{-modules over } \mathcal{S}_2\}$, taking $\mathfrak{a}$ to $\widehat{S}(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{LBA})$ is given by $\mu \mapsto \hbar\mu_\mathfrak{a}$, $\delta \mapsto \delta_\mathfrak{a}$.

1.11.4. *The prop* $\mathbf{LBA}_f$. Define $\widetilde{\mathrm{LBA}}_f$ as the prop with generators μ, f, δ and only relations: μ and δ satisfy the relations of LBA. Then $\widetilde{\mathrm{LBA}}_f$ has a grading $(\deg_\mu, \deg_\delta, \deg_f)$ by $\mathbb{N}^3$. For $F,G \in \mathrm{Ob}(\mathrm{Sch})$ and $x \in \widetilde{\mathrm{LBA}}_f(F,G)$ homogeneous, we have

$$|G| - |F| = \deg_\delta(x) - \deg_\mu(x) + 2\deg_f(x). \tag{11}$$

Then $\deg_\mu + \deg_\delta + 2\deg_f$ is a grading of LBA_f by $\mathbb{N}$. The corresponding filtration therefore satisfies condition (a). Since $\deg_\mu$, $\deg_\delta$ and $\deg_g$ are ≥ 0, (11) implies that it also satisfies conditions (b). Since the morphism $\widetilde{\mathrm{LBA}}_f$ to LBA_f is surjective, the filtration of $\widetilde{\mathrm{LBA}}_f$ induces a filtration of LBA_f satisfying (a) and (b). We denote by $\mathbf{LBA}_f$ the corresponding completion of LBA_f.

As before, if $\mathcal{LBA}_f$ is the category of pairs $(\mathfrak{a}, f_\mathfrak{a})$ of Lie bialgebras with twists, we have (a) a functor $\mathcal{LBA}_f \to \{S(\mathbf{LBA}_f)\text{-modules over } \mathcal{S}_1\}$, taking $(\mathfrak{a}, f_\mathfrak{a})$ to $S(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{LBA}_f)$ is given by $\mu \mapsto \mu_\mathfrak{a}$, $\delta \mapsto \hbar\delta_\mathfrak{a}$, $f \mapsto \hbar f_a$; and (b) a functor $\mathcal{LBA}_f \to \{S(\mathbf{LBA}_f)$-modules over $\mathcal{S}_2\}$, taking $(\mathfrak{a}, f_\mathfrak{a})$ to $\widehat{S}(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{LBA}_f)$ is given by $\mu \mapsto \hbar\mu_\mathfrak{a}$, $\delta \mapsto \delta_\mathfrak{a}$, $f \mapsto f_a$.

1.11.5. *The prop* **Cob**. Cob has a grading $(\deg_\mu, \deg_r)$ by $\mathbb{N}^2$. If $F,G \in \mathrm{Ob}(\mathrm{Sch})$ and $x \in \mathrm{Cob}(F,G)$ are homogeneous, then $|G| - |F| = 2\deg_r(x) - \deg_\mu(x)$. Then the $\mathbb{N}$-grading of Cob by $\deg_\mu + 2\deg_r$ induces a filtration satisfying (a) and (b). We denote the resulting topological prop by **Cob**.

If $\mathcal{C}ob$ is the category of coboundary Lie bialgebras $(\mathfrak{a}, r_\mathfrak{a})$, we have (a) a functor $\mathcal{C}ob \to \{S(\mathbf{Cob})\text{-modules over } \mathcal{S}_1\}$, taking $(\mathfrak{a}, r_\mathfrak{a})$ to $S(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{Cob})$ is given

by $\mu \mapsto \mu_{\mathfrak{a}}$, $r \mapsto \hbar r_a$; and (b) a functor $\mathcal{C}ob \to \{S(\mathbf{Cob})\text{-modules over } \mathcal{S}_2\}$, taking $(\mathfrak{a}, r_{\mathfrak{a}})$ to $\widehat{S}(\mathfrak{a})[[\hbar]]$; the representation of $S(\mathbf{Cob})$ is given by $\mu \mapsto \hbar\mu_{\mathfrak{a}}$, $r \mapsto r_a$.

1.11.6. *Morphisms between completed props.* The above morphisms Cob $\to$ Sch, LBA $\to$ Sch and $\mathrm{LBA}_f \to$ Sch are compatible with the filtrations, so they induce topological prop morphisms $\mathbf{Cob} \to \mathbf{Sch}$, $\mathbf{LBA} \to \mathbf{Sch}$ and $\mathbf{LBA}_f \to \mathbf{Sch}$.

Since κ_1 preserves the $\mathbb{N}$-grading, it extends to a morphism $\mathbf{LBA} \to \mathbf{LBA}_f$ of completed props.

κ_2 takes a monomial in (μ, δ) of bidegree (a, b) to a sum of monomials in (μ, δ, f) of degrees $(a+b'', b', b'')$, where $b'+b'' = b$. The $\mathbb{N}$-degree of such a monomial is $a+b'+3b'' \geq a+b$. So κ_2 preserves the descending filtrations of both props and extends to a morphism $\mathbf{LBA} \to \mathbf{LBA}_f$.

κ_0 takes a monomial in (μ, δ, f) either to 0 if the f-degree is > 0, or to the same monomial (which has the same $\mathbb{N}$-degree) otherwise. So κ_0 preserves the descending filtration and extends to a morphism $\mathbf{LBA}_f \to \mathbf{LBA}$.

Finally, κ takes a monomial in (μ, δ, f) of degree (a, b, c) (of $\mathbb{N}$-degree $a+b+2c$) to a monomial in (μ, ρ) of degree $(a+b, b+c)$, of $\mathbb{N}$-degree $a+3b+2c$. Since the $\mathbb{N}$-degree increases, κ preserves the descending filtration and extends to a morphism $\mathbf{LBA}_f \to \mathbf{Cob}$.

1.12. **The props P_α.** Let C be a coalgebra in **Sch**. This means that $C = \widehat{\oplus}_i C_i \in \mathbf{Sch}$ ($|C_i| = i$), and we have prop morphisms $C \to C^{\otimes 2}$, $C \to \mathbf{1}$ in **Sch**, such that the two morphisms $C \to C^{\otimes 3}$ coincide, and the composed morphisms $C \to C^{\otimes 2} \to C \otimes \mathbf{1} \simeq C$ and $C \to C^{\otimes 2} \to \mathbf{1} \otimes C \simeq C$ are the identity.

Let P be a prop. For $F = \widehat{\oplus}_i F_i$, $G = \widehat{\oplus}_i G_i$ in **Sch**, we set $P(F, G) = \oplus_{i,j} P(F_i, G_j)$. Then the operations $\circ : P(F, G) \otimes P(G, H) \to P(F, H)$ and $\otimes : P(F, G) \otimes P(F', G') \to P(F \otimes F', G \otimes G')$ are well-defined. Moreover, operations $\circ : P(F, G) \otimes \mathbf{Sch}(G, H) \to P(F, H)$ and $\circ : \mathbf{Sch}(F, G) \otimes P(G, H) \to P(F, H)$ are also well-defined.

We define a prop P_C by $P_C(F, G) := P(C \otimes F, G)$ for $F, G \in$ Sch. The composition of P_C is then defined as the map

$$P_C(F,G) \otimes P_C(G,H) \simeq P(C \otimes F, G) \otimes P(C \otimes G, H) \stackrel{(P(\mathrm{id}_C) \otimes -) \otimes \mathrm{id}}{\to}$$
$$P(C^{\otimes 2} \otimes F, C \otimes G) \otimes P(C \otimes G, H) \to P(C^{\otimes 2} \otimes F, H) \to P(C \otimes F, H) \simeq P_C(F, H)$$

and the tensor product is defined by

$$P_C(F,G) \otimes P_C(F',G') \simeq P(C \otimes F, G) \otimes P(C \otimes F', G') \stackrel{\otimes}{\to} P(C^{\otimes 2} \otimes F \otimes F', G \otimes G') \to P(C \otimes F \otimes F', G \otimes G').$$

We then have an isomorphism $P \simeq P_{\mathbf{1}}$ and a prop morphism $P \to P_C$ induced by $C \to \mathbf{1}$.

Let us define a P-coideal D of C to be the data of $D = \widehat{\oplus}_i D_i \in \mathbf{Sch}$, and morphisms $\alpha \in \widehat{\oplus}_i P(C_i, D)$, $\beta \in \widehat{\oplus}_i P(D_i, C \otimes D)$ and $\gamma \in \widehat{\oplus}_i P(D_i, D \otimes C)$, such that the diagrams

$$\begin{array}{ccc} C_i & \stackrel{\Delta_{|C_i}}{\to} & C^{\otimes 2} \\ {\scriptstyle \alpha_{|C_i}}\downarrow & & \downarrow{\scriptstyle \alpha \otimes \mathrm{id}_C} \\ D & \stackrel{\gamma}{\to} & D \otimes C \end{array} \quad \text{and} \quad \begin{array}{ccc} C_i & \stackrel{\Delta_{|C_i}}{\to} & C^{\otimes 2} \\ {\scriptstyle \alpha_{|C_i}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \alpha} \\ D & \stackrel{\beta}{\to} & C \otimes D \end{array}$$

commute for each i. A P-coideal D of C may be constructed as follows. Let $D' \in$ Sch, let $\alpha' \in P(C, D')$. Set $D := D' \otimes C$ and define $\alpha \in \widehat{\oplus}_i P(C_i, D)$ as the composed morphism $C \stackrel{\Delta}{\to} C^{\otimes 2} \stackrel{\alpha' \otimes \mathrm{id}}{\to} D' \otimes C = D$. We also define the morphism $\gamma \in \widehat{\oplus}_i P(D_i, D \otimes C)$ as the composition $D = D' \otimes C \stackrel{\mathrm{id} \otimes \Delta}{\to} D' \otimes C^{\otimes 2} = D \otimes C$ and $\beta \in \widehat{\oplus}_i \mathrm{LBA}(D, C \otimes D)$ as the composed morphism $D \to D \otimes C \to C \otimes D$.

If D is a P-coideal of C, set $P_D(F, G) := P(D \otimes F, G)$. Then for each (F, G), we have a morphism $P_D(F, G) \to P_C(F, G)$, such that the collection of all $\mathrm{Im}(P_D(F, G) \to P_C(F, G))$ is an ideal of P_C.

We denote by P_α as the corresponding quotient prop. We then have $P_\alpha(F,G) = \mathrm{Coker}(P_D(F,G) \to P_C(F,G))$ for any (F,G).

1.13. **Automorphisms of props.** For $\xi \in P(\mathbf{id},\mathbf{id})$, $\xi^{\otimes n} \in P(T_n,T_n) = \oplus_{\rho,\rho' \in \widehat{\mathfrak{S}}_n} \mathrm{Hom}(\pi_\rho,\pi_{\rho'}) \otimes P(Z_\rho,Z_{\rho'})$; as $\xi^{\otimes n}$ is $\mathfrak{S}_n^{diag}$-invariant ($\mathfrak{S}_n^{diag}$ being the diagonal subgroup of $\mathfrak{S}_n \times \mathfrak{S}_n$), we have $\xi^{\otimes n} = \oplus_{\rho \in \widehat{\mathfrak{S}}_n} \mathrm{id}_{\pi_\rho} \otimes \xi_\rho$, for some $\xi_\rho \in P(Z_\rho,Z_\rho)$. For $F = (F_\rho)_{\rho \in \sqcup_{n \geq 0} \widehat{\mathfrak{S}}_n}$, we set $\xi_F := \oplus_{\rho \in \sqcup_{n\geq 0} \widehat{\mathfrak{S}}_n} \mathrm{id}_{\pi_\rho} \otimes \xi_\rho \in P(F,F)$. One can prove that $\xi_{F \oplus G} = \xi_F \otimes \xi_G$, $\xi_{F \otimes G} = \xi_F \otimes \xi_G$, $(\xi \circ \eta)_F = \xi_F \circ \eta_F$. So if ξ is invertible, so are the ξ_F, and there is a unique prop automorphism $\theta(\xi)$ of P, taking $x \in P(F,G)$ to $\xi_G \circ x \circ \xi_F^{-1}$. The map $P(\mathbf{id},\mathbf{id})^\times \to \mathrm{Aut}(P)$ is a group morphism with normal image $\mathrm{Inn}(P)$. We call the elements of this image the inner automorphisms of P.

1.14. **Structure of the prop** LBA**.**

Lemma 1.6. *If $F,G \in \mathrm{Ob}(\mathrm{Sch})$, then we have an isomorphism*

$$\mathrm{LBA}(F,G) \simeq \oplus_{N \geq 0} (\mathrm{LCA}(F,T_N) \otimes \mathrm{LA}(T_N,G))_{\mathfrak{S}_N},$$

with inverse given by $f \otimes g \mapsto g \circ f$ (the prop morphisms LCA $\to$ LBA*,* LA $\to$ LBA *are understood).*

Proof. This has been proved in the case $F = T_n$, $G = T_m$ in [Enr2, Pos]. We then pass to the case of $F = Z_\rho$, $G = Z_\sigma$, $\rho \in \widehat{\mathfrak{S}}_n$, $\sigma \in \widehat{\mathfrak{S}}_m$ by identifying the isotypic components of this identity under the action of $\mathfrak{S}_n \times \mathfrak{S}_m$. The general case follows by linearity. □

According to (6), this result may be expressed as the isomorphism

$$\mathrm{LBA}(F,G) \simeq \oplus_{Z \in \mathrm{Irr}(\mathrm{Sch})} \mathrm{LCA}(F,Z) \otimes \mathrm{LA}(Z,G). \tag{12}$$

We also have:

Lemma 1.7. *If A,B are finite sets, then $\mathrm{LA}(T_A,T_B) = \oplus_{f:A\to B \text{ surjective}} \otimes_{b \in B} \mathrm{LA}(T_{f^{-1}(b)},T_{\{b\}})$, where the inverse map is given by the tensor product.*

We now prove:

Lemma 1.8. *Let $F_1,...,F_n,G_1,...,G_p \in \mathrm{Ob}(\mathrm{Sch})$, then we have a decomposition*

$$\mathrm{LBA}(\otimes_{i=1}^n F_i, \otimes_{j=1}^p G_j) = \bigoplus_{(Z_{ij})_{i,j} \in \mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}} \mathrm{LBA}((F_i)_i,(G_j)_j)_{(Z_{ij})_{i,j}},$$

where

$$\mathrm{LBA}((F_i)_i,(G_j)_j)_{(Z_{ij})_{i,j}} = \bigotimes_{i=1}^n \mathrm{LCA}(F_i, \otimes_{\beta=1}^p Z_{i\beta})\big) \otimes \big(\bigotimes_{j=1}^p \mathrm{LA}(\otimes_{\alpha=1}^n Z_{\alpha j}, G_j),$$

with inverse given by $(\otimes_i f_i) \otimes (\otimes_j g_j) \mapsto (\otimes_j g_j) \circ \sigma_{n,p} \circ (\otimes_i f_i)$; here $\sigma_{n,p}$ is the braiding isomorphism $\otimes_i(\otimes_j Z_{ij}) \to \otimes_j(\otimes_i Z_{ij})$.

Proof. The l.h.s. is equal to

$$\oplus_{N\geq 0} \oplus_{(n_{ij})|\sum_{i,j} n_{ij} = N} \big((\otimes_i \mathrm{LCA}(F_i, T_{\sum_j n_{ij}})) \otimes (\otimes_j \mathrm{LA}(T_{\sum_i n_{ij}}, G_j))\big)_{\prod_{i,j} \mathfrak{S}_{n_{ij}}}.$$

(6) then implies the result. □

1.15. **Structure of the prop** LBA_f**.** In the construction of Subsection 1.12, we set $C := S\circ\wedge^2$, $D' = \wedge^3$, and $\alpha' \in \mathrm{LBA}(C, D') = \oplus_{k\geq 0}\mathrm{LBA}(S^k \circ \wedge^2, \wedge^3)$ has only nonzero components for $k = 1, 2$; for $k = 1$, this component specializes to $\wedge^2(\mathfrak{a}) \to \wedge^3(\mathfrak{a})$,

$$f \mapsto (\delta \otimes \mathrm{id})(f) + (\delta \otimes \mathrm{id})(f)^{231} + (\delta \otimes \mathrm{id})(f)^{312}$$

and for $k = 2$ it specializes to $S^2(\wedge^2(\mathfrak{a})) \to \wedge^3(\mathfrak{a})$,

$$f^{\otimes 2} \mapsto [f^{12}, f^{13}] + [f^{12}, f^{23}] + [f^{13}, f^{23}].$$

Then $\alpha : C \to D$ is a LBA-coideal. We denote by LBA_α the corresponding quotient prop P_α.

Proposition 1.9. *There exists a prop isomorphism* $\mathrm{LBA}_f \xrightarrow{\sim} \mathrm{LBA}_\alpha$.

Proof. Using the presentation of LBA_f, one checks that there is a unique prop morphism $\mathrm{LBA}_f \to \mathrm{LBA}_\alpha$, taking μ to the class of $\mu \in \mathrm{LBA}(\wedge^2, \mathbf{id}) \subset \oplus_{k\geq 0}\mathrm{LBA}((S^k \circ \wedge^2) \otimes \wedge^2, \mathbf{id}) = \mathrm{LBA}((S\circ\wedge^2)\otimes\wedge^2, \mathbf{id})$, taking δ to the class of $\delta \in \mathrm{LBA}(\mathbf{id}, \wedge^2) \subset \oplus_{k\geq 0}\mathrm{LBA}((S^k\circ\wedge^2)\otimes\mathbf{id}, \wedge^2) = \mathrm{LBA}((S \circ \wedge^2) \otimes \mathbf{id}, \wedge^2)$, and taking f to the class of $\mathrm{id}_{\wedge^2} \in \mathrm{LBA}(\wedge^2, \wedge^2) \subset \oplus_{k\geq 0}\mathrm{LBA}(S^k \circ \wedge^2, \wedge^2) = \mathrm{LBA}((S \circ \wedge^2) \otimes \mathbf{1}, \wedge^2)$.

We now construct a prop morphism $\mathrm{LBA}_\alpha \to \mathrm{LBA}_f$.

We construct a linear map $\mathrm{LBA}((S^k \circ \wedge^2) \otimes F, G) \to \mathrm{LBA}_f(F, G)$ as follows: using the prop morphism $\mathrm{LBA} \to \mathrm{LBA}_f$ given by $\mu, \delta \mapsto \mu, \delta$, we get a linear map $\mathrm{LBA}((S^k \circ \wedge^2) \otimes F, G) \to \mathrm{LBA}_f((S^k \circ \wedge^2) \otimes F, G)$. We have an element $f^{\otimes k} \in \mathrm{LBA}_f(S^k \circ \mathbf{1}, S^k \circ \wedge^2)$ so the operation $x \mapsto x \circ (f^{\otimes k} \otimes \mathrm{id}_F)$ is a linear map

$$\mathrm{LBA}_f((S^k \circ \wedge^2) \otimes F, G) \to \mathrm{LBA}_f((S^k \circ \mathbf{1}) \otimes F, G) \simeq \mathrm{LBA}_f(F, G).$$

The composition of these maps is a linear map $\mathrm{LBA}((S^k\circ\wedge^2)\otimes F, G) \to \mathrm{LBA}_f(F, G)$. Summing up these maps, we get a linear map $\mathrm{LBA}((S \circ \wedge^2) \otimes F, G) \to \mathrm{LBA}_f(F, G)$, and one checks that it factors through a linear map $\mathrm{LBA}_\alpha(F, G) \to \mathrm{LBA}_f(F, G)$. One also checks that this map is compatible with the prop operations, so it is a prop morphism.

We now show that the composed morphisms $\mathrm{LBA}_f \to \mathrm{LBA}_\alpha \to \mathrm{LBA}_f$ and $\mathrm{LBA}_\alpha \to \mathrm{LBA}_f \to \mathrm{LBA}_\alpha$ are both the identity.

In the case of $\mathrm{LBA}_f \to \mathrm{LBA}_\alpha \to \mathrm{LBA}_f$, one shows that the composed map takes each generator of LBA_f to itself, hence is the identity.

Let us show that $\mathrm{LBA}_\alpha \to \mathrm{LBA}_f \to \mathrm{LBA}_\alpha$ is the identity. We already defined the prop $\widetilde{\mathrm{LBA}}_f$. Then we have a canonical prop morphism $\widetilde{\mathrm{LBA}}_f \to \mathrm{LBA}_f$. We also have prop morphisms $\mathrm{LBA}_{S\circ\wedge^2} \to \widetilde{\mathrm{LBA}}_f$ and $\widetilde{\mathrm{LBA}}_f \to \mathrm{LBA}_{S\circ\wedge^2}$, defined similarly to $\mathrm{LBA}_\alpha \to \mathrm{LBA}_f$ and $\mathrm{LBA}_f \to \mathrm{LBA}_\alpha$. We then have commuting squares

$$\begin{array}{ccc} \mathrm{LBA}_{S\circ\wedge^2} & \to & \widetilde{\mathrm{LBA}}_f \\ \downarrow & & \downarrow \\ \mathrm{LBA}_\alpha & \to & \mathrm{LBA}_f \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathrm{LBA}}_f & \to & \mathrm{LBA}_{S\circ\wedge^2} \\ \downarrow & & \downarrow \\ \mathrm{LBA}_f & \to & \mathrm{LBA}_\alpha \end{array}$$

One checks that the composed morphism $\mathrm{LBA}_{S\circ\wedge^2} \to \widetilde{\mathrm{LBA}}_f \to \mathrm{LBA}_{S\circ\wedge^2}$ is the identity, which implies that $\mathrm{LBA}_\alpha \to \mathrm{LBA}_f \to \mathrm{LBA}_\alpha$ is the identity. $\square$

In what follows, we will use the above isomorphism to identify LBA_α with LBA_f. The main output of this identification is the construction of a grading of $\mathrm{LBA}_f(\otimes_i F_i, \otimes_j G_j)$ by families (Z_{ij}) in $\mathrm{Irr}(\mathrm{Sch})$, since as we now show, props of the form LBA_α all give rise to such a grading.

Let $C, D \in \mathrm{Ob}(\mathbf{Sch})$ and let $\alpha \in \widehat{\oplus}_i\mathrm{LBA}(C_i, D)$.

Proposition 1.10. *Let* $F_1, ..., F_n, G_1, ..., G_p \in \mathrm{Ob}(\mathrm{Sch})$. *Set* $F := \otimes_i F_i$, $G := \otimes_j G_j$. *For* $Z = (Z_{ij})_{i\in[n], j\in[p]}$ *a map* $[n] \times [p] \to \mathrm{Irr}(\mathrm{Sch})$, *set*

$$\mathrm{LBA}_C((F_i)_i, (G_j)_j)_Z := \oplus_{\tilde Z\in\mathrm{Irr}(\mathrm{Sch})^{(\{0\}\cup[n])\times[p]} \,|\tilde Z_{|[n]\times[p]}=Z}\mathrm{LBA}([C, (F_i)_i], (G_j)_j)_{\tilde Z},$$

where $[C,(F_i)_i] \in \mathrm{Ob}(\mathrm{Sch})^{\{0\}\cup[n]}$ *is the extension of* $(F_i)_i$ *defined by* $0 \mapsto C$.

Then $\mathrm{LBA}_C(F,G) = \oplus_{Z\in \mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}} \mathrm{LBA}_C((F_i)_i,(G_j)_j)_Z$.

Moreover, the map $\mathrm{LBA}_D(F,G) \to \mathrm{LBA}_C(F,G)$ *preserves the grading by* $\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}$. *The cokernel of this map therefore inherits a grading*

$$\mathrm{LBA}_\alpha(F,G) = \oplus_{Z\in \mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}} \mathrm{LBA}_\alpha((F_i)_i,(G_j)_j)_Z.$$

Proof. The first statement follows from Lemma 1.8 (with $F_0 = C$). Let us prove the second statement. Consider the sequence of maps

$$\mathrm{LCA}(D, \otimes_{j\in[p]} Z_{0j}) \stackrel{\alpha\otimes -}{\to} \mathrm{LBA}(C,D)\otimes \mathrm{LBA}(D,\otimes_{j\in[p]}Z_{0j}) \stackrel{\circ}{\to} \mathrm{LBA}(D,\otimes_{j\in[p]}Z_{0j})$$
$$\simeq \oplus_{(Z'_{0j})_j\in\mathrm{Irr}(\mathrm{Sch})^{[p]}} \mathrm{LCA}(C,\otimes_{j\in[p]}Z'_{0j})\otimes(\otimes_{j\in[p]}\mathrm{LA}(Z'_{0j},Z_{0j})),$$

$$\kappa \mapsto \oplus_{(Z'_{0j})_j\in\mathrm{Irr}(\mathrm{Sch})^{[p]}} \sum_\alpha \kappa'_\alpha \otimes (\otimes_{j\in[p]}\lambda'_{j,\alpha}),$$

where the first map uses the prop morphism $\mathrm{LCA}\to\mathrm{LBA}$.

For $\tilde Z \in \mathrm{Irr}(\mathrm{Sch})^{(\{0\}\cup[n])\times[p]}$, $\tilde Z_{|[n]\times[p]}$ is its restriction to $[n]\times[p]$. The map $\mathrm{LBA}(D\otimes F,G)\simeq \mathrm{LBA}_D(F,G)\to\mathrm{LBA}_C(F,G)$ restricts to

$$\mathrm{LBA}([D,(F_i)_i],(G_j)_j)_{\tilde Z} \to \mathrm{LBA}_C((F_i)_i,(G_j)_j)_{\tilde Z_{|[n]\times[p]}}$$

$$(\otimes_{j\in[p]}\lambda_j)\circ\sigma_{n+1,p}\circ(\kappa\otimes(\otimes_{i\in[n]}\kappa_i)) \mapsto \oplus_{(Z'_{0j})_j\in\mathrm{Irr}(\mathrm{Sch})^{[p]}} \sum_\alpha \Big(\otimes_{j\in[p]}\big(\lambda_j\circ(\lambda'_{j,\alpha}\otimes \mathrm{id}_{\otimes_{i\in[n]}Z_{ij}})\big)\Big)\circ\sigma_{n+1,p}\circ\kappa'_\alpha.$$

Summing up over $(Z_{0j})_j \in \mathrm{Irr}(\mathrm{Sch})^{[p]}$, we get the result. □

1.16. **Partial traces on Π^0_{LBA}, $\Pi^0_{\mathrm{LBA}_f}$.** Recall that for $F,G\in\mathrm{Ob}(\mathrm{Sch}_{(1)})$, $\Pi^0_{\mathrm{LBA}}(F,G) = \mathrm{LBA}(c(F),c(G))$. For $F,G\in\mathrm{Ob}(\mathrm{Sch}_{p,q})$, we introduce a grading of $\Pi^0_{\mathrm{LBA}}(F,G)$ by $\mathcal{G}_0([p],[q])$ as follows. Assume first that F,G are simple, so $F = \boxtimes_{i=1}^n Z_{\rho_i}$, $G = \boxtimes_{j=1}^p Z_{\sigma_p}$. If $Z = (Z_{ij})_{i,j}\in \mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}$, we define the support of Z as $\mathrm{supp}(Z) := \{(i,j)|Z_{ij}\neq \mathbf{1}\}$. Then

$$\Pi^0_{\mathrm{LBA}}(\boxtimes_{i=1}^n Z_{\rho_i},\boxtimes_{j=1}^p Z_{\sigma_j})_S := \oplus_{Z\in\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}\,|\,\mathrm{supp}(Z)=S} \mathrm{LBA}((Z_{\rho_i}),(Z_{\sigma_j}))_Z.$$

If $F = (F_{\rho_1,\ldots,\rho_n})$, $G = (G_{\sigma_1,\ldots,\sigma_p})$,

$$\Pi^0_{\mathrm{LBA}}(F,G)_S := \oplus_{(\rho_i),(\sigma_j)} \mathrm{Vect}(F_{(\rho_i)},G_{(\sigma_j)})\otimes\Pi^0_{\mathrm{LBA}}(\boxtimes_{i=1}^n Z_{\rho_i},\boxtimes_{j=1}^p Z_{\sigma_j})_S.$$

Proposition 1.11. *This grading is compatible with the monoidal category structure of* $\mathcal{G}_0$, *namely: for* $F,G,H\in\mathrm{Ob}(\mathrm{Sch}_{p,q,r})$,

$$\Pi^0_{\mathrm{LBA}}(G,H)_{S'}\circ\Pi^0_{\mathrm{LBA}}(F,G)_S \subset \Pi^0_{\mathrm{LBA}}(F,H)_{S'\circ S},$$

for $F_i,G_i\in\mathrm{Ob}(\mathrm{Sch}_{p_i,q_i})$,

$$\Pi^0_{\mathrm{LBA}}(F_1,G_1)_{S_1}\boxtimes\Pi^0_{\mathrm{LBA}}(F_2,G_2)_{S_2}\subset\Pi^0_{\mathrm{LBA}}(F_1\boxtimes F_2,G_1\boxtimes G_2)_{S_1\otimes S_2}$$

(here $\boxtimes$ *denotes the tensor product operation of* Π^0_{LBA}*), and* $\beta_{F,G}\in\Pi^0_{\mathrm{LBA}}(F\boxtimes G,G\boxtimes F)_{\beta_{[n],[p]}}$.

Proof. The only nontrivial statement is the first one. Let Z, Z' be such that $\mathrm{supp}(Z) = S$, $\mathrm{supp}(Z') = S'$. For F, G, H simple, the composition factors as

$$\Pi^0_{\mathrm{LBA}}(F,G)_Z \otimes \Pi^0_{\mathrm{LBA}}(G,H)_{Z'}$$
$$\to (\otimes_i \mathrm{LCA}(Z_{\rho_i}, \otimes_j Z_{ij})) \otimes (\otimes_j \mathrm{LA}(\otimes_i Z_{ij}, Z_{\sigma_j})) \otimes (\otimes_j \mathrm{LCA}(Z_{\sigma_j}, \otimes_k Z'_{jk})) \otimes (\otimes_k \mathrm{LA}(\otimes_j Z'_{jk}, Z_{\tau_k}))$$
$$\to \oplus_{(Z''_{ijk})} (\otimes_i \mathrm{LCA}(Z_{\rho_i}, \otimes_j Z_{ij})) \otimes (\otimes_{j,i} \mathrm{LCA}(Z_{ij}, \otimes_k Z''_{ijk}))$$
$$\otimes (\otimes_{j,k} \mathrm{LA}(\otimes_i Z''_{ijk}, Z'_{jk})) \otimes (\otimes_k \mathrm{LA}(\otimes_j Z'_{jk}, Z_{\tau_k}))$$
$$\to \oplus_{(Z''_{ijk})} (\otimes_i \mathrm{LA}(Z_{\rho_i}, \otimes_{j,k} Z''_{ijk})) \otimes (\otimes_j \mathrm{LCA}(\otimes_{i,j} Z''_{ijk}, Z_{\sigma_j}))$$
$$\simeq \oplus_{(Z''_{ijk})} (\otimes_i \mathrm{LA}(Z_{\rho_i}, \otimes_k Z'''_{ik})) \otimes (\otimes_j \mathrm{LCA}(\otimes_i Z'''_{ik}, Z_{\sigma_j})) \to \Pi^0_{\mathrm{LBA}}(F,H),$$

where the first map is the decomposition map, the second map is the tensor product over j of the jth exchange map (composition followed by decomposition) $\mathrm{LA}(\otimes_i Z_{ij}, Z_{\sigma_j}) \otimes \mathrm{LCA}(Z_{\sigma_j}, \otimes_k Z'_{jk}) \to \mathrm{LBA}(\otimes_i Z_{ij}, \otimes_k Z'_{jk}) \to \oplus_{(Z''_{ijk})} (\otimes_i \mathrm{LCA}(Z_{ij}, \otimes_k Z''_{ijk})) \otimes (\otimes_k \mathrm{LA}(\otimes_i Z''_{ijk}, Z'_{jk}))$, the third map is composition in LA and LCA, the fourth map is obtained by $Z'''_{ik} := \otimes_j Z''_{ijk}$, the fifth map is composition.

If $(i,k) \in S' \circ S$, then for some $j \in J$, $(i,j) \in S$ and $(j,k) \in S'$; so Z_{ij}, Z'_{jk} are $\neq \mathbf{1}$. So the component of the target of the jth exchange map corresponding to $(i,k) \mapsto Z'''_{ijk} = \mathbf{1}$, is zero; hence the component of the composition of the 3 first maps of the diagram, coresponding to $(i,k) \mapsto Z'''_{ijk} = \mathbf{1}$, is zero. It follows that Z''_{ik} in the forelast vector space are sums of objects of $\mathrm{Irr}(\mathrm{Sch})$ with degree > 0.

On the other hand, if $(i,k) \notin S' \circ S$, then for any $j \in J$, Z_{ij} or $Z'_{jk} = \mathbf{1}$. Then the component of the target of the jth exchange map corresponding to any $(i,k) \mapsto Z'''_{ijk}$ different from $(i,k) \mapsto \mathbf{1}$, is zero; si Z''_{ik} is the forelast vector space are sums of copies of $\mathbf{1}$.

It follows that the image of the overall map is contained in $\Pi_{\mathrm{LBA}}(F,G)_{S' \circ S}$. $\square$

Let $F, G, H \in \mathrm{Ob}(\mathrm{Sch}_{p,q,r})$, and let us define the diagram

$$\Pi^0_{\mathrm{LBA}}(F \boxtimes H, G \boxtimes H) \supset \Pi^0_{\mathrm{LBA}}(F, G|H) \stackrel{\mathrm{tr}_H}{\to} \Pi^0_{\mathrm{LBA}}(F,G)$$

(the general case is then derived by linearity). Let us first assume that F, G, H are simple, so $F = Z_{(\rho_i)} := \boxtimes_{i \in I} Z_{\rho_i}$, $G = Z_{(\sigma_j)} = \boxtimes_{j \in J} Z_{\sigma_j}$, $H = Z_{(\tau_k)} = \boxtimes_{k \in K} Z_{\tau_k}$ (I, J, K are ordered sets of cardinality p, q, r, and $(\rho_i), (\sigma_j), (\tau_k)$ are maps $I, J, K \to \sqcup_{n \geq 0} \widehat{\mathfrak{S}}_n$). Recall that $\Pi^0_{\mathrm{LBA}}(F \boxtimes H, G \boxtimes H) = \oplus_{S \in \mathcal{G}_0 \Pi^0_{\mathrm{LBA}}(I \otimes K, J \otimes K)} (F \boxtimes H, G \boxtimes H)_S$. We then set

$$\Pi^0_{\mathrm{LBA}}(F, G|H) := \oplus_{S \in \mathcal{G}_0(I, J|K)} \Pi^0_{\mathrm{LBA}}(F \boxtimes H, G \boxtimes H)_S.$$

We then define the linear map

$$\mathrm{tr}_H : \Pi^0_{\mathrm{LBA}}(F, G|H)_S = \oplus_{Z |\, \mathrm{supp}(Z) = S} \Pi^0_{\mathrm{LBA}}(F \boxtimes H, G \boxtimes H)_Z \to \Pi^0_{\mathrm{LBA}}(F,G)_{\mathrm{tr}_K(S)}$$

as follows. Recall from section 1.3 the order relation $\prec$ on K, the total order relation $<$ on K, its extension to a relation on $I \sqcup K \sqcup J$, the numbering $K = \{k_1, ..., k_{|K|}\}$, the sets K_α.

Let $Z = (Z_{uv})_{(u,v) \in (I \sqcup K) \times (K \sqcup J)}$ be such that $\mathrm{supp}(Z) = S$. For $k \in K$, set $H_k := \boxtimes_{x \in K_k} Z(x)$, where $Z(k) := Z_{\tau_k}$, $Z(u,v) := Z_{\rho_u \sigma_v}$ for $(u,v) \in (I \sqcup K) \times (K \sqcup J)$ (we extend ρ to $I \sqcup K$ by $\rho_k := \tau_k$ and σ to $K \sqcup J$ by $\sigma_k := \tau_k$). Set also $H_{k,k+1} := Z_{k,k+1} \boxtimes (\boxtimes_{(u,v) \in K'_{\alpha,\alpha+1} \sqcup K''_{\alpha,\alpha+1} \sqcup K_{\alpha,\alpha+1}} Z(u,v))$.

Set also $H_0 := F$, $H_{01} := Z(k_1) \boxtimes (\boxtimes_{(u,v) \in I \times (K \sqcup J), u \prec v} Z(u,v))$, and $H_{|K|+1} = G$, $H_{|K|,|K|+1} = Z(k_{|K|}) \boxtimes (\boxtimes_{(u,v) \in (I \sqcup K) \times J} Z(u,v))$.

Then tr_H is the sum over Z with $\mathrm{supp}(Z)=S$ of the compositions

$$\Pi^0_{\mathrm{LBA}}(F\boxtimes H,G\boxtimes H)_Z=(\otimes_{u\in I\sqcup K}\mathrm{LCA}(Z_{\rho_u},\otimes_{v\in K\sqcup J}Z_{uv}))\otimes(\otimes_{v\in K\sqcup J}\mathrm{LA}(\otimes_{u\in I\sqcup K}Z_{uv},Z_{\sigma_v}))$$
$$\to\bigotimes_{k=0}^{|K|}\Pi^0_{\mathrm{LCA}}(H_k,H_{k,k+1})\otimes\Pi^0_{\mathrm{LA}}(H_{k,k+1},H_{k+1})\stackrel{\circ}{\to}\Pi^0_{\mathrm{LBA}}(F,G)\qquad(13)$$

(one checks that this map is independent on the ordering of K).

The sum of these maps takes $\Pi^0_{\mathrm{LBA}}(F\boxtimes H,G\boxtimes H)_S$ to $\otimes_{k=0}^{|K|}\Pi^0_{\mathrm{LBA}}(H_k,H_{k+1})_{S_{K_k,K_{k+1}}}$, where $K_0:=I$, $K_{k+1}:=J$ (with the notation of Section 1.3), therefore the image of the above map is contained in $\Pi^0_{\mathrm{LBA}}(F,G)_{\mathrm{tr}_K(S)}$.

If now F,G,H are arbitrary elements of $\mathrm{Ob}(\mathrm{Sch}_{p,q,r})$, namely $F=(F_{\rho_1,\ldots,\rho_p})$, $G=(G_{\sigma_1,\ldots,\sigma_q})$, $H=(H_{\tau_1,\ldots,\tau_r})$, then

$$\Pi^0_{\mathrm{LBA}}(F\boxtimes H,G\boxtimes H)=\oplus_{(\rho_i),(\sigma_j),(\tau_k),(\tau'_k)}\mathrm{Vect}(F_{(\rho_i)}\otimes H_{(\tau_k)},G_{(\sigma_j)}\otimes H_{(\tau'_k)})\otimes\Pi^0_{\mathrm{LBA}}(Z_{(\rho_i)}\boxtimes Z_{(\tau_k)},Z_{(\sigma_j)}\boxtimes Z_{(\tau'_k)});$$

here $Z_{(\rho_i)}=\boxtimes_{i\in I}Z_{\rho_i}$, etc. Then $\Pi^0_{\mathrm{LBA}}(F,G|H)$ is homogeneous w.r.t. this decomposition; its components for $(\tau_k)\neq(\tau'_k)$ coincide with those of $\Pi^0_{\mathrm{LBA}}(F\boxtimes H,G\boxtimes H)$ and the component for $(\tau'_k)=(\tau_k)$ is $\mathrm{Vect}(F_{(\rho_i)}\otimes H_{(\tau_k)},G_{(\sigma_j)}\otimes H_{(\tau_k)})\otimes\Pi^0_{\mathrm{LBA}}(Z_{(\rho_i)},Z_{(\sigma_j)}|Z_{(\tau'_k)})$; the restriction of tr_H to a components of the first kind is 0 and its restriction to the last component is the tensor product the partial trace with $\mathrm{tr}_{(Z_{\tau_k})}$. Then one checks that the diagrams $\Pi^0_{\mathrm{LBA}}(F\boxtimes H,G\boxtimes H)\supset\Pi^0_{\mathrm{LBA}}(F,G|H)\to\Pi^0_{\mathrm{LBA}}(F,G)$ define a partial trace on Π^0_{LBA}.

Let us define now a partial trace on $\Pi^0_{\mathrm{LBA}_f}$ (and more generally on the $\Pi^0_{\mathrm{LBA}_\alpha}$). Let C be a coalgebra in **Sch**, we have for $F,G\in\mathrm{Ob}(\mathrm{Sch}_{p,q})$, $\Pi^0_{\mathrm{LBA}_C}(F,G)=\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)$. A partial trace is then defined on $\Pi^0_{\mathrm{LBA}_C}$ as follows: $\Pi^0_{\mathrm{LBA}_C}(F,G|H)\subset\Pi^0_{\mathrm{LBA}_C}(F\boxtimes H,G\boxtimes H)$ is $\Pi^0_{\mathrm{LBA}_C}(F,G|H):=\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G|H)$ and $\Pi^0_{\mathrm{LBA}_C}(F,G|H)\stackrel{\mathrm{tr}_H}{\to}\Pi^0_{\mathrm{LBA}_C}(F,G)$ coincides with $\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G|H)\stackrel{\mathrm{tr}_H}{\to}\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)$. One checks that this defines a partial trace on $\Pi^0_{\mathrm{LBA}_C}$.

If $Z\in\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}$, we set $\Pi^0_{\mathrm{LBA}_C}(F,G)_Z:=\oplus_{\tilde Z}\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)_{\tilde Z}$, where the sum is over the $\tilde Z:(\{0\}\sqcup[n])\times[p]\to\mathrm{Irr}(\mathrm{Sch})$, such that $\tilde Z_{|[n]\times[p]}=Z$. We also set, for $S\subset[n]\times[p]$, $\Pi^0_{\mathrm{LBA}_C}(F,G)_S:=\oplus_{Z|\,\mathrm{supp}(Z)=S}\Pi^0_{\mathrm{LBA}_C}(F,G)_Z$. Then:

Lemma 1.12. *The properties of* Π^0_{LBA} *extend to* $\Pi^0_{\mathrm{LBA}_C}$*, namely* $\Pi^0_{\mathrm{LBA}_C}(G,H)_{S'}\circ\Pi^0_{\mathrm{LBA}_C}(F,G)_S\subset\Pi^0_{\mathrm{LBA}_C}(F,H)_{S'\circ S}$, $\Pi^0_{\mathrm{LBA}_C}(F,G)_S\boxtimes\Pi^0_{\mathrm{LBA}_C}(F',G')_{S'}\subset\Pi^0_{\mathrm{LBA}_C}(F\boxtimes F',G\boxtimes G')_{S\otimes S'}$, $\beta_{F,G}\in\Pi^0_{\mathrm{LBA}_C}(F\boxtimes G,G\boxtimes F)_{\beta_{[n],[p]}}$*, and for* $S\in\mathcal{G}_0(I,J|K)$, $\mathrm{tr}_H(\Pi^0_{\mathrm{LBA}_C}(F,G|H)_S)\subset\Pi^0_{\mathrm{LBA}_C}(F,G)_{\mathrm{tr}_K(S)}$.

Proof. $\Pi^0_{\mathrm{LBA}_C}(F,G)_S=\oplus_{\tilde S\in(\{0'\}\sqcup[n])\times[p]|\tilde S\cap([n]\times[p])=S}\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)_{\tilde S}$. In the same way, $\Pi^0_{\mathrm{LBA}_C}(G,H)_S=\oplus_{\tilde S'\in(\{0''\}\sqcup[p])\times[q]|\tilde S'\cap([p]\times[q])=S'}\Pi^0_{\mathrm{LBA}}(C\boxtimes G,H)_{S'}$. Then for $\tilde S\subset(\{0'\}\sqcup[n])\times[p]$, $\tilde S'\subset(\{0''\}\sqcup[p])\times[q]$, let $\tilde S'*\tilde S:=\{(i,k)\in(\{0',0''\}\sqcup[n])\times p|(i,k)\in(\{0'\}\sqcup[n])\times[q]$ and there exists $k\in[p]$ with $(i,j)\in S$, $(j,k)\in S'$ or $i=0''$ and $(i,k)\in S'\}$. Then the composition $\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)\otimes\Pi^0_{\mathrm{LBA}}(C\boxtimes G,H)\to\Pi^0_{\mathrm{LBA}}(C\boxtimes F,H)$ maps $\Pi^0_{\mathrm{LBA}}(C\boxtimes F,G)_{\tilde S}\otimes\Pi^0_{\mathrm{LBA}}(C\boxtimes G,H)_{\tilde S'}$ to $\Pi^0_{\mathrm{LBA}}(C\boxtimes F,H)_{(\tilde S'*\tilde S)\circ\emptyset}\oplus\Pi^0_{\mathrm{LBA}}(C\boxtimes F,H)_{(\tilde S'*\tilde S)\circ\Delta_{0,0'0''}}$, where $\emptyset,\Delta_{0,0'0''}\in\mathcal{G}_0(\{0\}\sqcup[n],\{0',0''\}\sqcup[n])$ are $\emptyset$ and $\Delta_{00'}=\{(0,0'),(0,0'')\}$. Now both $(\tilde S'*\tilde S)\circ\emptyset$ and $(\tilde S'\circ\tilde S)\circ\Delta_{0,0'0''}$ are elements of $\mathcal{G}_0(\{0\}\sqcup[n],[q])$, with their intersection with $[n]\times[p]$ equal to $S'\circ S$. This proves the first statement. The other statements are proved in the same way. $\square$

If $D\in\mathrm{Ob}(\mathbf{Sch})$, we similarly set $\Pi^0_D(F,G):=\Pi^0_{\mathrm{LBA}}(D\boxtimes F,G)$. The diagram $\Pi^0_D(F\boxtimes H,G\boxtimes H)\supset\Pi^0_D(F,G|H)\stackrel{\mathrm{tr}_H}{\to}\Pi^0_D(F,G)$ is defined as above. We define $\Pi^0_D(F,G)_Z$ and $\Pi^0_D(F,G)_S$ as above. The following properties generalize to this more general setup: $\beta_{F,G}\in\Pi^0_D(F\boxtimes G,G\boxtimes F)_{\beta_{[n],[p]}}$, and for $S\in\mathcal{G}_0(I,J|K)$, $\mathrm{tr}_H(\Pi^0_D(F,G|H)_S)\subset\Pi^0_D(F,G)_{\mathrm{tr}_K(S)}$.

Lemma 1.13. *$\alpha \in \widehat{\oplus}_i \mathrm{LBA}(C_i, D)$ induces a linear map $\Pi^0_D(F,G) \to \Pi^0_{\mathrm{LBA}_C}(F,G)$, which is compatible with the gradings by $\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}$ (and therefore also by $\mathcal{G}_0([n],[p])$). Then $\Pi^0_{\mathrm{LBA}_\alpha}(F,G) = \mathrm{Coker}(\Pi^0_D(F,G) \to \Pi^0_{\mathrm{LBA}_C}(F,G))$, and $\Pi^0_{\mathrm{LBA}_\alpha}(F,G) = \oplus_{S\in\mathcal{G}_0(I,J)}\Pi_{\mathrm{LBA}_\alpha}(F,G)_S$.*

For each $S \in \mathcal{G}_0(I,J|K)$, the diagram

$$\begin{array}{ccc} \Pi^0_D(F\boxtimes H, G\boxtimes H)_S & \stackrel{\mathrm{tr}_H}{\to} & \Pi^0_D(F,G)_{\mathrm{tr}_K(S)} \\ \downarrow & & \downarrow \\ \Pi^0_{\mathrm{LBA}_C}(F\boxtimes H, G\boxtimes H)_S & \stackrel{\mathrm{tr}_H}{\to} & \Pi^0_{\mathrm{LBA}_C}(F,G)_{\mathrm{tr}_K(S)} \end{array}$$

commutes; its vertical cokernel is a linear map $\mathrm{tr}_H : \Pi^0_{\mathrm{LBA}_\alpha}(F\boxtimes H, G\boxtimes H)_S \to \Pi^0_{\mathrm{LBA}_\alpha}(F,G)_{\mathrm{tr}_K(S)}$. We set $\Pi^0_{\mathrm{LBA}_\alpha}(F,G|H) := \oplus_{S\in\mathcal{G}_0(I,J|K)}\Pi^0_{\mathrm{LBA}_\alpha}(F\boxtimes H, G\boxtimes H)_S$, then we have a diagram $\Pi^0_{\mathrm{LBA}_\alpha}(F\boxtimes H, G\boxtimes H) \supset \Pi^0_{\mathrm{LBA}_\alpha}(F,G|H) \stackrel{\mathrm{tr}_H}{\to} \Pi^0_{\mathrm{LBA}_\alpha}(F,G)$.

If $\alpha : C \to D$ is a LBA-coideal, then the multiprop $\Pi^0_{\mathrm{LBA}_\alpha}$ is graded by $\mathcal{G}_0$, and (tr_H) is a partial trace on $\Pi^0_{\mathrm{LBA}_\alpha}$, compatible with this grading.

Proof. The first statement is a consequence of Proposition 1.10. The commutativity of the diagram follows from the fact that for $x \in \Pi^0_{\mathrm{LBA}_D}(F,G|H)$, $x\circ(\alpha\boxtimes\mathrm{id}_{F\boxtimes H}) \in \Pi^0_{\mathrm{LBA}_C}(F,G|H)$ and $\mathrm{tr}_H(x\circ(\alpha\boxtimes\mathrm{id}_{F\boxtimes H})) = \mathrm{tr}_H(x)\circ(\alpha\boxtimes\mathrm{id}_F)$. The remaining properties follow from those of $\Pi^0_{\mathrm{LBA}_C}$. □

Remark 1.14. One also checks that the partial trace on $\mathcal{G}_0$, as well as its counterparts on $\Pi^0_{\mathrm{LBA}_\alpha}$, have the following cyclicity properties. If $S \in \mathcal{G}_0(U\otimes I, V'\otimes J)$ and $S' \in \mathcal{G}_0(V\otimes J, U'\otimes I)$, then $S'\circ(\beta_{V,V'}\otimes\mathrm{id}_J)\circ S \in \mathcal{G}_0(V\otimes U, V'\otimes U'|I)$ iff $S\circ(\beta_{U,U'}\otimes\mathrm{id}_I)\circ S' \in \mathcal{G}_0(U\otimes V, U'\otimes V'|J)$, and we then have $\mathrm{tr}_I(S'\circ(\beta_{V,V'}\otimes\mathrm{id}_J)\circ S) = \beta_{V',U'}\circ\mathrm{tr}_J(S\circ(\beta_{U,U'}\otimes\mathrm{id}_I)\circ S')\circ\beta_{U,V}$. If now S, S' are as above, $F_U = \boxtimes_{u\in I}Z_{\rho_u}$, etc., and $x \in \Pi^0_{\mathrm{LBA}}(F_U\boxtimes F_I, F_{V'}\boxtimes F_J)_S$, $x' \in \Pi^0_{\mathrm{LBA}}(F_V\boxtimes F_J, F_{U'}\boxtimes F_I)_{S'}$, then $x'\circ(\beta_{F_V,F_{V'}}\boxtimes\mathrm{id}_{F_J})\circ x \in \Pi^0_{\mathrm{LBA}}(F_V\boxtimes F_U, F_{V'}\boxtimes F_{U'}|F_I)$, $x\circ(\beta_{F_U,F_{U'}}\boxtimes\mathrm{id}_{F_I})\circ x' \in \Pi_{\mathrm{LBA}}(F_U\boxtimes F_V, F_{U'}\boxtimes F_{V'}|F_J)$, and

$$\mathrm{tr}_{F_I}(x'\circ(\beta_{F_V,F_{V'}}\otimes\mathrm{id}_{F_J})\circ x) = \beta_{F_{V'},F_{U'}}\circ\mathrm{tr}_{F_J}(x\circ(\beta_{F_U,F_{U'}}\otimes\mathrm{id}_{F_I})\circ x')\circ\beta_{F_U,F_V}.$$

□

1.17. **Morphisms of multiprops with partial traces.** The prop morphisms $\kappa_{1,2} : \mathrm{LBA} \to \mathrm{LBA}_f$, $\kappa_0 : \mathrm{LBA}_f \to \mathrm{LBA}$ and $\tau : \mathrm{LBA} \to \mathrm{LBA}$ induce morphisms between the corresponding multiprops $\Pi^0_{\mathrm{LBA}_f}$ and Π^0_{LBA} (still denoted κ_i, etc.).

We will prove:

Proposition 1.15. *These morphisms intertwine the traces.*

Proof. Let $\kappa : \mathrm{LBA}_\alpha \to \mathrm{LBA}_\beta$ be any for these morphisms. We will prove that for any simple $F,G,H \in \mathrm{Ob}(\mathrm{Sch}_{I,J,K})$, $\kappa(\Pi^0_{\mathrm{LBA}_\alpha}(F,G|H)) \subset \Pi^0_{\mathrm{LBA}_\beta}(F,G|H)$, and then that the diagram

$$\begin{array}{ccc} \Pi^0_{\mathrm{LBA}_\alpha}(F,G|H) & \stackrel{\kappa}{\to} & \Pi^0_{\mathrm{LBA}_\beta}(F,G|H) \\ {\scriptstyle \mathrm{tr}_H}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_H} \\ \Pi^0_{\mathrm{LBA}_\alpha}(F,G) & \stackrel{\kappa}{\to} & \Pi^0_{\mathrm{LBA}_\beta}(F,G) \end{array}$$

commutes.

The case of τ is clear. In the case of κ_0, we argue as follows: let $C : S\circ\wedge^2$, $D := \wedge^3\otimes(S\circ\wedge^2)$, then C is a coalgebra in **Sch** and $\alpha : C \to D$ is a LBA-coideal in C. The coalgebra morphism $\mathbf{1} \to C$ induces a multiprop morphism $\Pi^0_{\mathrm{LBA}_C} \to \Pi^0_{\mathrm{LBA}_{\mathbf{1}}} = \Pi^0_{\mathrm{LBA}}$; the composed morphism $\Pi^0_{\mathrm{LBA}_D} \to \Pi^0_{\mathrm{LBA}_C} \to \Pi^0_{\mathrm{LBA}}$ so we get a multiprop morphism $\Pi^0_{\mathrm{LBA}_f} \to \Pi^0_{\mathrm{LBA}}$, compatible with the traces. The maps $\Pi^0_{\mathrm{LBA}_f}(F,G) \to \Pi^0_{\mathrm{LBA}}(F,G)$ are the maps induced by κ_0, which proves the statement in the case of κ_0.

The coalgebra morphism $C \to \mathbf{1}$ induces a morphism of multiprops $\Pi^0_{\mathrm{LBA}} \simeq \Pi^0_{\mathrm{LBA}_{\mathbf{1}}} \to \Pi_{\mathrm{LBA}_C}$, compatible with the traces, which we compose with the projection $\Pi_{\mathrm{LBA}_C} \to \Pi_{\mathrm{LBA}_f}$. The maps $\Pi^0_{\mathrm{LBA}}(F,G) \to \Pi^0_{\mathrm{LBA}_f}(F,G)$ are the maps induced by κ_1, which proves the statement in the case of κ_1.

We now treat the case of κ_2. For $F = \boxtimes_{i\in I} Z_{\rho_i}$, $G = \boxtimes_{j\in J} Z_{\sigma_j}$, where $(\rho_i)_i$, $(\sigma_j)_j$ are maps $I, J \to \sqcup_{n\geq 0}\widehat{\mathfrak{S}}_n$. For $Z \in \mathrm{Irr}(\mathrm{Sch})^{I\times J}$, set

$$\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_Z := (\otimes_{i\in I}\mathrm{LBA}(Z_{\rho_i}, \otimes_{j\in J} Z_{ij})) \otimes (\otimes_{j\in J}\mathrm{LBA}(\otimes_{i\in I} Z_{ij}, Z_{\sigma_j}))$$

and for $S \in \mathcal{G}_0(I,J)$,

$$\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_S := \oplus_{Z|\,\mathrm{supp}(Z)=S}\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_Z.$$

The operations of LBA (tensor products, composition, braidings) give rise to a natural map $\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_Z \to \Pi^0_{\mathrm{LBA}}(F,G)$, which add up to a map $\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_S \to \Pi^0_{\mathrm{LBA}}(F,G)$.

Lemma 1.16. *The image of this map is equal to* $\Pi^0_{\mathrm{LBA}_\alpha}(F,G)_S$.

Proof. This image contains $\Pi^0_{\mathrm{LBA}}(F,G)_S$, as $\Pi^0_{\mathrm{LBA}}(F,G)_S$ is the subspace of $\Pi^0_{\mathrm{LBA}}(F,G)_S$, where the successive LBA are replaced by LCA, LA. Let us prove the opposite inclusion.

For each Z, the map $\underline{\Pi}^0_{\mathrm{LBA}}(F,G)_Z \to \Pi^0_{\mathrm{LBA}}(F,G)$ factors as

$$\begin{aligned}
&(\otimes_{i=1}^n \mathrm{LBA}(Z_{\rho_i}, \otimes_j Z_{ij})) \otimes (\otimes_{j=1}^p \mathrm{LBA}(\otimes_i Z_{ij}, Z_{\sigma_j}))\\
&\to \oplus_{Z',Z''\in\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}}(\otimes_i \mathrm{LCA}(Z_{\rho_i}, \otimes_j Z'_{ij})) \otimes (\otimes_{i,j}\mathrm{LA}(Z'_{ij}, Z_{ij})) \otimes (\otimes_{i,j}\mathrm{LCA}(Z_{ij}, Z''_{ij}))\\
&\otimes (\otimes_j \mathrm{LA}(\otimes_i Z''_{ij}, Z_{\sigma_j}))\\
&\to \oplus_{Z',Z'',Z'''\in\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}}(\otimes_i \mathrm{LCA}(Z_{\rho_i}, \otimes_j Z'_{ij})) \otimes (\otimes_{i,j}\mathrm{LCA}(Z'_{ij}, Z'''_{ij})) \otimes (\otimes_{i,j}\mathrm{LA}(Z'''_{ij}, Z''_{ij}))\\
&\otimes (\otimes_j \mathrm{LA}(\otimes_i Z''_{ij}, Z_{\sigma_j}))\\
&\to \oplus_{Z'''\in\mathrm{Irr}(\mathrm{Sch})^{[n]\times[p]}}(\otimes_i \mathrm{LCA}(Z_{\rho_i}, \otimes_j Z'''_{ij}))(\otimes_j \mathrm{LA}(\otimes_i Z'''_{ij}, Z_{\sigma_j})) \to \Pi^0_{\mathrm{LBA}}(\boxtimes_i Z_{\rho_j}, \boxtimes_j Z_{\sigma_j}),
\end{aligned}$$

where the first map is a tensor product of decompositions of $\mathrm{LBA}(\otimes_i F_i, \otimes_j G_j)$, the second map is a tensor product of exchange maps (composition followed by decomposition) $\mathrm{LA}(Z'_{ij}, Z_{ij}) \otimes \mathrm{LCA}(Z_{ij}, Z''_{ij}) \to \mathrm{LBA}(Z'_{ij}, Z''_{ij}) \to \oplus_{Z'''_{ij}}\mathrm{LCA}(Z'_{ij}, Z'''_{ij})\otimes\mathrm{LA}(Z''_{ij}, Z''_{ij})$, the third map is a tensor product of compositions in LA and LCA. Now if $Z'_{ij} = \mathbf{1}$ (resp., $\neq \mathbf{1}$), the components of the exchange map corresponding to any $Z'''_{ij} \neq \mathbf{1}$ (resp., $Z'''_{ij} = \mathbf{1}$), are zero. Therefore the components of the composition of the three first maps, where $\mathrm{supp}(Z''') \neq \mathrm{supp}(Z)$, are zero. It follows that if $\mathrm{supp}(Z) = S$, the image of the overall map is contained in $\Pi^0_{\mathrm{LBA}}(\boxtimes_i Z_{\rho_i}, \boxtimes_j Z_{\sigma_j})_S$, as wanted. □

We also define $\underline{\Pi}^0_{\mathrm{LBA}_C}(F,G)_Z := \oplus_{Z'|Z'_{|I\times J}}\Pi^0_{\mathrm{LBA}}(C\boxtimes F, G)$, and $\underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)_Z$ as the image of $\underline{\Pi}^0_{\mathrm{LBA}_C}(F,G)_Z$ in $\underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)$. We define similarly $\underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)_S$ for $S \in \mathcal{G}_0(I,J)$. Arguing as in the above Lemma, one show that the image of the natural map $\underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)_S \to \Pi^0_{\mathrm{LBA}_f}(F,G)$ is contained in $\Pi^0_{\mathrm{LBA}_f}(F,G)_S$.

Lemma 1.17. *The map* $\kappa_2 : \Pi^0_{\mathrm{LBA}}(F,G)_S \to \Pi^0_{\mathrm{LBA}_f}(F,G)$ *factors through* $\Pi^0_{\mathrm{LBA}}(F,G)_S \to \underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)_S \to \Pi^0_{\mathrm{LBA}_f}(F,G)_S$, *so it is compatible with the gradings by* $\mathcal{G}_0(I,J)$.

Proof of Lemma. For each $Z \in \mathrm{Irr}(\mathrm{Sch})^{I\times J}$, the restriction of this map factors as

$$\begin{aligned}
&\Pi^0_{\mathrm{LBA}}(F,G)_Z = (\otimes_{i\in I}\mathrm{LCA}(Z_{\rho_i}, \otimes_{j\in J} Z_{ij})) \otimes (\otimes_{j\in J}\mathrm{LA}(\otimes_{i\in I} Z_{ij}, Z_{\sigma_j}))\\
&\stackrel{\kappa_2}{\to} (\otimes_{i\in I}\mathrm{LBA}_f(Z_{\rho_i}, \otimes_{j\in J} Z_{ij})) \otimes (\otimes_{j\in J}\mathrm{LBA}_f(\otimes_{i\in I} Z_{ij}, Z_{\sigma_j})) \to \Pi^0_{\mathrm{LBA}_f}(F,G).
\end{aligned}$$

The statement follows from the fact that the image of the last map is contained in $\Pi^0_{\mathrm{LBA}_f}(F,G)_Z$. □

If $H = \boxtimes_{k\in K} Z_{\tau_k}$, and $S \in \mathcal{G}_0(I,J|K)$, then we define

$$\underline{\mathrm{tr}}_H : \underline{\Pi}^0_{\mathrm{LBA}_f}(F,G|H)_S \to \Pi^0_{\mathrm{LBA}_f}(F,G)_{\mathrm{tr}_K(S)}$$

similarly to tr_H (see (13), where $\mathrm{LCA}, \mathrm{LA}$ are replaced by LBA and Π^0_{LCA}, Π^0_{LA} are replaced by Π^0_{LBA}).

Lemma 1.18. *The diagram*

$$\begin{array}{ccc}
\underline{\Pi}^0_{\mathrm{LBA}_f}(F,G|H)_S & & \\
\downarrow & \searrow \underline{\mathrm{tr}}_H & \\
\Pi^0_{\mathrm{LBA}_f}(F,G|H)_S & \xrightarrow{\mathrm{tr}_H} & \Pi^0_{\mathrm{LBA}_f}(F,G)_{\mathrm{tr}_K(S)}
\end{array}$$

commutes.

Proof. We first prove that the similar diagram commutes, where LBA_f is replaced by LBA. For $Z = (Z_{uv})_{(u,v)\in(I\sqcup K)\times(K\sqcup J)}$ such that $\mathrm{supp}(Z) = S$, the vertical map restricts to

$$\begin{aligned}
&\underline{\Pi}^0_{\mathrm{LBA}}(F,G|H)_Z = (\otimes_{i\in I\sqcup K}\mathrm{LBA}(Z_{\rho_i}, \otimes_{j\in J\sqcup K} Z_{ij})) \otimes (\otimes_{j\in K\sqcup J}\mathrm{LBA}(\otimes_{i\in I\sqcup K} Z_{ij}, Z_{\sigma_j}))\\
&\simeq \oplus_{Z_{iij},Z_{ijj}\in\mathrm{Irr(Sch)}}[\otimes_{i\in I\sqcup K}\{\mathrm{LCA}(Z_{\rho_i}, \otimes_{j\in K\sqcup J} Z_{iij}) \otimes \otimes_{j\in J\sqcup K}\mathrm{LA}(Z_{iij}, Z_{ij})\}]\\
&\otimes [\otimes_{j\in K\sqcup J}\{\otimes_{i\in I\sqcup K}\mathrm{LCA}(Z_{ij}, Z_{ijj}) \otimes \mathrm{LA}(\otimes_{i\in I\sqcup K} Z_{ijj}, Z_{\sigma_j})\}]\\
&\to \oplus_{Z_{iij},Z_{ijj},Z'_{ij}\in\mathrm{Irr(Sch)}}[\otimes_{i\in I\sqcup K}\{\mathrm{LCA}(Z_{\rho_i}, \otimes_{j\in K\sqcup J} Z_{iij}) \otimes \otimes_{j\in J\sqcup K}\mathrm{LCA}(Z_{iij}, Z'_{ij})\}]\\
&\otimes [\otimes_{j\in K\sqcup J}\{\otimes_{i\in I\sqcup K}\mathrm{LA}(Z'_{ij}, Z_{ijj}) \otimes \mathrm{LA}(\otimes_{i\in I\sqcup K} Z_{ijj}, Z_{\sigma_j})\}]\\
&\to \oplus_{Z'_{ij}\in\mathrm{Irr(Sch)}}[\otimes_{i\in I\sqcup K}\mathrm{LCA}(Z_{\rho_i}, \otimes_{j\in K\sqcup J} Z'_{ij})] \otimes [\otimes_{j\in K\sqcup J}\mathrm{LA}(\otimes_{i\in I\sqcup K} Z'_{ij}, Z_{\sigma_j})]\\
&\simeq \Pi^0_{\mathrm{LBA}}(F,G|H)_S
\end{aligned}$$

Define $(H_k)_k$ and $(H_{k,k+1})_k$ as above; define also $(H'_k)_k$ and $(H'_{k,k+1})_k$ similarly, replacing (Z_{ij}) by (Z'_{ij}), and $(H''_{k,k+1})_k, (H'''_{k,k+1})_k$ by replacing (Z_{ij}) by (Z_{iij}), (Z_{ijj}).

Then each square of the following diagram commutes

$$\begin{array}{ccc}
\begin{array}{c}(\otimes_{i\in I\sqcup K}\mathrm{LBA}(Z_{\rho_i},\otimes_{j\in J\sqcup K} Z_{ij}))\\ \otimes(\otimes_{j\in K\sqcup J}\mathrm{LBA}(\otimes_{i\in I\sqcup K} Z_{ij}, Z_{\sigma_j}))\end{array} & \to & \otimes_{k=0}^{|K|}\Pi^0_{\mathrm{LBA}}(H_k,H_{k+1})\otimes\Pi^0_{\mathrm{LBA}}(H_{k,k+1},H_{k+1})\\
\downarrow & & \downarrow\\
\begin{array}{c}\oplus_{Z_{iij},Z_{ijj}}[\otimes_{i\in I\sqcup K}\{\mathrm{LCA}(Z_{\rho_i},\otimes_{j\in K\sqcup J} Z_{iij})\\ \otimes\otimes_{j\in J\sqcup K}\mathrm{LA}(Z_{iij},Z_{ij})\}]\\ \otimes[\otimes_{j\in K\sqcup J}\{\otimes_{i\in I\sqcup K}\mathrm{LCA}(Z_{ij},Z_{ijj})\\ \otimes\mathrm{LA}(\otimes_{i\in I\sqcup K} Z_{ijj},Z_{\sigma_j})\}]\end{array} & \to & \begin{array}{c}\oplus_{Z_{iij},Z_{ijj}}\otimes_{k=0}^{|K|}\Pi^0_{\mathrm{LCA}}(H_k,H''_{k,k+1})\otimes\Pi^0_{\mathrm{LA}}(H''_{k,k+1},H_{k,k+1})\\ \otimes\Pi^0_{\mathrm{LCA}}(H_{k,k+1},H'''_{k,k+1})\otimes\Pi^0_{\mathrm{LA}}(H'''_{k,k+1},H_{k+1})\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{c}\oplus_{Z_{iij},Z_{ijj},Z'_{ij}\in\mathrm{Irr(Sch)}}\\ [\otimes_{i\in I\sqcup K}\{\mathrm{LCA}(Z_{\rho_i},\otimes_{j\in K\sqcup J} Z_{iij})\otimes\\ \otimes_{j\in J\sqcup K}\mathrm{LCA}(Z_{iij},Z'_{ij})\}]\\ \otimes[\otimes_{j\in K\sqcup J}\{\otimes_{i\in I\sqcup K}\mathrm{LA}(Z'_{ij},Z_{ijj})\\ \otimes\mathrm{LA}(\otimes_{i\in I\sqcup K} Z_{ijj},Z_{\sigma_j})\}]\end{array} & \to & \begin{array}{c}\oplus_{Z_{iij},Z_{ijj},Z'_{ij}}\otimes_{k=0}^{|K|}\Pi^0_{\mathrm{LCA}}(H_k,H''_{k,k+1})\otimes\Pi^0_{\mathrm{LCA}}(H''_{k,k+1},H'_{k,k+1})\\ \otimes\Pi^0_{\mathrm{LA}}(H'_{k,k+1},H'''_{k,k+1})\otimes\Pi^0_{\mathrm{LA}}(H'''_{k,k+1},H_{k+1})\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{c}\oplus_{Z'_{ij}\in\mathrm{Irr(Sch)}}[\otimes_{i\in I\sqcup K}\mathrm{LCA}(Z_{\rho_i},\otimes_{j\in K\sqcup J} Z'_{ij})]\\ \otimes[\otimes_{j\in K\sqcup J}\mathrm{LA}(\otimes_{i\in I\sqcup K} Z'_{ij},Z_{\sigma_j})]\end{array} & \to & \oplus_{Z'_{ij}}\otimes_{k=0}^{|K|}\Pi^0_{\mathrm{LCA}}(H_k,H'_{k,k+1})\otimes\Pi^0_{\mathrm{LA}}(H'_{k,k+1},H_{k+1})\\
\downarrow & & \downarrow\\
\Pi^0_{\mathrm{LBA}}(F,G|H)_S & \to & \Pi^0_{\mathrm{LBA}}(F,G)
\end{array}$$

which implies the commutativity in the case of Π^0_{LBA}. The proof is the same in the case of $\Pi^0_{\mathrm{LBA}_f}$. $\square$

End of proof of Proposition 1.15. The proposition now follows from the commutativity of the above diagram, together with that of

$$\begin{array}{ccc} \Pi^0_{\mathrm{LBA}}(F,G|H)_S & \stackrel{\mathrm{tr}_H}{\to} & \Pi^0_{\mathrm{LBA}}(F,G)_{\mathrm{tr}_K(S)} \\ \downarrow & & \downarrow \\ \underline{\Pi}^0_{\mathrm{LBA}_f}(F,G|H)_S & \stackrel{\mathrm{tr}_H}{\to} & \underline{\Pi}^0_{\mathrm{LBA}_f}(F,G)_{\mathrm{tr}_K(S)} \end{array}$$

□

1.18. **The quasi-bi-multiprops Π and Π_f.** Let Π, Π_f be the quasi-bi-multiprops associated to the multiprops with traces Π^0_{LBA}, $\Pi^0_{\mathrm{LBA}_f}$, and the involution $F \mapsto F^*$ of $\mathrm{Sch}_{(1)}$. Explicitly, we have $\Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G') := \mathrm{LBA}(c(F)\otimes c(G')^*, c(F')\otimes c(G)^*)$ and $\Pi_f(F\underline{\boxtimes}G, F'\underline{\boxtimes}G') := \mathrm{LBA}_f(c(F)\otimes c(G')^*, c(F')\otimes c(G)^*)$.

Since $\kappa_{1,2} : \Pi^0_{\mathrm{LBA}} \to \Pi^0_{\mathrm{LBA}_f}$, $\kappa_0 : \Pi^0_{\mathrm{LBA}} \to \Pi^0_{\mathrm{LBA}}$ and $\tau : \Pi^0_{\mathrm{LBA}} \to \Pi^0_{\mathrm{LBA}}$ are morphisms of multiprops with traces, they induce morphisms $\kappa^{\Pi}_{1,2}$, etc., between the corresponding quasi-bi-multiprops.

We now define a degree on Π as follows. For $F \in \mathrm{Ob}(\mathrm{Sch}_{(1)})$ of the form $F = \boxtimes_{i\in[n]} Z_{\rho_i}$, we set $|F| = \sum_{i\in[n]} |Z_{\rho_i}|$. For $F,...,G' \in \mathrm{Ob}(\mathrm{Sch}_{(1)})$ and $x \in \Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G') = \mathrm{LBA}(c(F)\otimes c(G')^*, c(F')\otimes c(G)^*)$ homogeneous, we set $\deg_\Pi(x) := \deg_\delta(x) + |G'| - |G|$. If $x \in \Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')_Z = \Pi^0_{\mathrm{LBA}}(F\boxtimes (G')^*, F'\boxtimes G^*)_Z$, then $\deg_\Pi(x) = \sum_{(s,t)\in([n]\sqcup[m'])\times([n']\sqcup[m])} |Z_{st}| - |F| - |G|$, so $\deg_\Pi(x) \geq 0$. One checks that $\deg_\Pi$ is a degree on Π, i.e., it is additive under composition and tensor product.

We now define a degree on Π_f. We first define a degree on LBA_f as follows: f and δ have degree 1 and μ has degree 0. If now $x \in \Pi_f(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')$, we set $\deg_{\Pi_f}(x) := \deg_{\mathrm{LBA}_f}(x) + |G'| - |G|$. Then $\deg_{\Pi_f}(x) \geq 0$, and $\deg_{\Pi_f}$ defines a degree on Π_f

We define completions of Π and Π_f as follows: for $B, B' \in \mathrm{Ob}(\mathbf{Sch}_{(1+1)})$, $\mathbf{\Pi}(B,B')$ (resp., $\mathbf{\Pi}_f(B,B')$) is the degree completion of $\Pi(B,B')$ (resp., $\Pi_f(B,B')$). The morphisms $\kappa^{\Pi}_{1,2}$ of quasi-bi-multiprops are of degree 0, and induce therefore morphisms between their completions.

It follows from the cyclicity of the trace on $\mathcal{G}_0$ that we have an involution of $\mathcal{G}$, defined as follows: it acts on objects by $(I,J) \mapsto (J,I)$, and on morphisms by $\mathcal{G}((I,J),(I',J')) = \mathcal{G}_0(I\sqcup J', I'\sqcup J) \ni x \mapsto \beta_{I,J'} \circ x \circ \beta_{J,I'} \in \mathcal{G}_0(J'\sqcup I, J\sqcup I') \in \mathcal{G}((J',I'),(J,I))$.

The cyclicity of the trace on Π^0_{LBA} implies that the bi-multiprop Π is equipped with a compatible involution, described as follows: its acts on objects as $F\underline{\boxtimes}G \mapsto G^*\underline{\boxtimes}F^*$, and on morphisms by $\Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G') = \mathrm{LBA}(c(F)\otimes c(G')^*, c(F')\otimes c(G)^*) \ni x \mapsto \beta_{c(F'),c(G^*)} \circ x \circ \beta_{c(G'^*),c(F)} \in \mathrm{LBA}(c(G^*)\otimes c(F'), c(G'^*)\otimes c(F)) = \Pi(G^*\underline{\boxtimes}F^*, (G')^*\underline{\boxtimes}(F')^*)$. This involution has degree zero, hence extends to $\mathbf{\Pi}$.

If $B \in \mathrm{Ob}(\mathrm{Sch}_{(1+1)})$, we define $\mathrm{can}_B \in \Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, B\boxtimes B^*)$ as follows. If $B = Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p}$, where $Z_{\rho_1,...,\rho_n} = (\boxtimes_{i=1}^n Z_{\rho_i})$, then $B\boxtimes B^* = (Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p}) \boxtimes (Z_{\sigma_1^*,...,\sigma_p^*}\underline{\boxtimes}Z_{\rho_1^*,...,\rho_n^*}) = Z_{\rho_1,...,\sigma_p^*}\underline{\boxtimes}Z_{\sigma_1,...,\rho_n^*}$, so $\Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, Z_{\rho_1,...,\sigma_p^*}\underline{\boxtimes}Z_{\sigma_1,...,\rho_n^*}) = \Pi^0_{\mathrm{LBA}}(Z^*_{\sigma_1,...,\rho_n^*}, Z_{\rho_1,...,\sigma_p^*}) = \Pi^0_{\mathrm{LBA}}(Z_{\sigma_1^*,...,\sigma_n^*}\boxtimes Z_{\rho_1,...,\rho_n}, Z_{\rho_1,...,\rho_n}\boxtimes Z_{\sigma_1^*,...,\sigma_n^*})$, and can_B corresponds to $\beta_{Z_{\rho_1,...,\rho_n}, Z_{\sigma_1^*,...,\sigma_n^*}}$. If now $B = (B_{\rho_1,...,\rho_n;\sigma_1,...,\sigma_p})$, we set $\mathrm{can}_B = \oplus \mathrm{id}_{B_{\rho_1,...,\sigma_p}} \otimes \mathrm{can}_{Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p}} \in \oplus \mathrm{End}(B_{\rho_1,...,\sigma_p}) \otimes \Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p})\boxtimes (Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p})^*) \subset \Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, B\boxtimes B^*)$.

We define similarly $\mathrm{can}^*_B \in \Pi(B\boxtimes B^*, \mathbf{1}\underline{\boxtimes}\mathbf{1})$ as follows. If $B = Z_{\rho_1,...,\rho_n}\underline{\boxtimes}Z_{\sigma_1,...,\sigma_p}$, then $\Pi(B\boxtimes B^*, \mathbf{1}\underline{\boxtimes}\mathbf{1}) = \Pi^0_{\mathrm{LBA}}(Z_{\rho_1,...,\sigma_p^*}, Z^*_{\sigma_1,...,\rho_n^*})$ an can^*_B corresponds to $\beta_{Z_{\rho_1,...,\rho_n}, Z_{\sigma_1^*,...,\sigma_p^*}}$; we then extend this definition by linearity as above.

The involution of Π can then be described as follows: for $x \in \Pi(B,C)$, $x^* \in \Pi(C^*,B^*)$ can be expressed as $x^* = (\mathrm{can}^*_C \boxtimes \mathrm{id}_{B^*}) \circ (x \boxtimes \beta_{B^*,C^*}) \circ (\mathrm{can}_B \boxtimes \mathrm{id}_{C^*})$.

The quasi-bi-multiprops Π, Π_f give rise to quasi-biprops π, π_f by the inclusion $\mathrm{Ob}(\mathrm{Sch}_{1+1}) \subset \mathrm{Ob}(\mathrm{Sch}_{(1+1)})$. Their topological versions $\mathbf{\Pi}, \mathbf{\Pi}_f$ give rise to topological quasi-biprops $\boldsymbol{\pi}, \boldsymbol{\pi}_f$.

We define sub-bimultiprops $\Pi^{\text{left,right}}$ and $\Pi_f^{\text{left,right}}$ as follows. For $F = \boxtimes_{i\in I} Z_{\rho_i}$, $G = \boxtimes_{j\in J} Z_{\sigma_j}$, etc., we set $\Pi^{\text{left,right}}(F\underline{\boxtimes}G, F'\underline{\boxtimes}G') := \oplus_{S\in\mathcal{G}^{\text{left,right}}\Pi((I,J),(I',J'))}(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')_S$, and a similar definition in the case of Π_f. These bimultiprops have topological versions $\mathbf{\Pi}, \mathbf{\Pi}_f^{\text{left,right}}$.

The sub-bimultiprops give rise to sub-biprops $\pi^{\text{left,right}}$ of π and $\pi_f^{\text{left,right}}$ of π_f, as well as to topological sub-biprops $\boldsymbol{\pi}^{\text{left,right}}$ and $\boldsymbol{\pi}_f^{\text{left,right}}$ of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_f$.

1.19. **Cokernels in** LA**.** Let $\mathrm{i}_{\mathrm{LA}} \in \mathbf{LA}(T\otimes T_2\otimes T, T)$ be the prop morphism $m_T^{(2)}\circ\big(\mathrm{id}_T\otimes\big((12)-(21)-\mu\big)\otimes\mathrm{id}_T\big)$, where $m_T \in \mathbf{Sch}(T^{\otimes 2}, T)$ is the propic version of the product in the tensor algebra and $m_T^{(2)} \in \mathbf{Sch}(T^{\otimes 3}, T)$ is its 2-fold iterate.

Let $p_{\mathrm{LA}} \in \mathbf{LA}(T, S)$ be the direct sum, for $n \geq 0$, of $p_{\mathrm{LA},n} \in \mathbf{LA}(\mathbf{id}^{\otimes n}, S)$ given by $m_{\mathrm{PBW}}^{(n)} \circ \mathrm{inj}_1^{\boxtimes n}$, where $m_{\mathrm{PBW}} \in \mathbf{LA}(S^{\otimes 2}, S)$ is the propic version of the PBW star-product,[4] $m_{\mathrm{PBW}}^{(n)} = m_{\mathrm{PBW}} \circ \ldots \circ (m_{\mathrm{PBW}} \boxtimes \mathrm{id}_{S^{\otimes n-2}}) \in \mathbf{LA}(S^{\otimes n}, S)$ and $\mathrm{inj}_1 : \mathbf{id} \to S$ is the canonical morphism.

Then the composed morphism $p_{\mathrm{LA}} \circ \mathrm{i}_{\mathrm{LA}}$ is zero. Moreover, one proves using the image of $\mathrm{sym} \in \mathbf{Sch}(S, T) \to \mathbf{LA}(S, T)$, where sym is the symmetrization map, that $T \overset{p_{\mathrm{LA}}}{\to} S$ is a cokernel for $T \otimes T_2 \otimes T \overset{\mathrm{i}_{\mathrm{LA}}}{\to} T$.

The following diagram then commutes in **LA**:

$$\begin{array}{ccc} T^{\otimes 2} & \overset{m_T}{\to} & T \\ \downarrow & & \downarrow \\ S^{\otimes 2} & \overset{m_{\mathrm{PBW}}}{\to} & S \end{array}$$

Let $\Delta_0^S \in \mathbf{Sch}(S, S^{\otimes 2}) \subset \mathbf{LA}(S, S^{\otimes 2})$ be the propic version of the coproduct of the symmetric algebra $S(V)$, where V is a vector space. Then

$$\Delta_0^S \circ m_{\mathrm{PBW}} = (m_{\mathrm{PBW}} \otimes m_{\mathrm{PBW}}) \circ (1324) \circ \Big(\Delta_0^S \otimes \Delta_0^S\Big); \tag{14}$$

In the same way, $T^{\otimes n} \overset{p_{\mathrm{LA}}^{\otimes n}}{\to} S^{\otimes n}$ is a cokernel for $\oplus_{i=0}^{n-1} T^{\otimes i} \otimes \Big(T \otimes T_2 \otimes T\Big) \otimes T^{\otimes n-1-i} \to T^{\otimes n}$.

Observe that LA is not an abelian category, as some morphisms (e.g., $\mu \in \mathrm{LA}(\wedge^2, \mathbf{id})$) do not admit cokernels.

1.20. **The element $\delta_S \in \mathbf{LBA}(S, S^{\otimes 2})$.** There is a unique element $\delta_T \in \mathbf{LBA}(T, T^{\otimes 2})$, such that $\delta_T \circ \mathrm{inj}_1 = \mathrm{can} \circ \delta$, where $\mathrm{inj}_1 : \mathbf{id} \to T$ and $\mathrm{can} : \wedge^2 \hookrightarrow \mathbf{id}^{\otimes 2} \hookrightarrow T^{\otimes 2}$ are the canonical injections, and such that

$$\delta_T \circ m_T = m_T^{\otimes 2} \circ (1324) \circ (\delta_T \otimes \Delta_0^T + \Delta_0^T \otimes \delta_T),$$

where $\Delta_0^T \in \mathbf{Sch}(T, T^{\otimes 2})$ is the propic version of the coproduct of $T(V)$, where V is primitive. There exists $\delta_{T\otimes T_2\otimes T} \in \mathbf{LBA}(T \otimes T_2 \otimes T, T \otimes (T \otimes T_2 \otimes T) \oplus (T \otimes T_2 \otimes T) \otimes T)$, such that the diagram

$$\begin{array}{ccc} T \otimes T_2 \otimes T & \overset{\delta_{T\otimes T_2\otimes T}}{\to} & T \otimes (T \otimes T_2 \otimes T) \oplus (T \otimes T_2 \otimes T) \otimes T \\ \mathrm{i}_{\mathrm{LA}}\downarrow & & \downarrow \mathrm{id}_T \otimes \mathrm{i}_{\mathrm{LA}} + \mathrm{i}_{\mathrm{LA}} \otimes \mathrm{id}_T \\ T & \overset{\delta_T}{\to} & T \otimes T \end{array}$$

commutes. Taking cokernels, we get a morphism $\delta_S \in \mathbf{LBA}(S, S^{\otimes 2})$, such that

$$\delta_S \circ m_{\mathrm{PBW}} = m_{\mathrm{PBW}}^{\otimes 2} \circ (1324) \circ \big(\delta_S \otimes \Delta_0^S + \Delta_0^S \otimes \delta_S\big),$$

and $\delta_S \circ \mathrm{inj}_1 = \mathrm{can} \circ \delta$, where $\mathrm{can} : \wedge^2 \subset \mathbf{id}^{\otimes 2} \subset S^{\otimes 2}$ is the canonical inclusion. We also have $((12) + (21)) \circ \delta_S = 0$.

[4]The PBW star-product is the map $S(\mathfrak{a})^{\otimes 2} \to S(\mathfrak{a})$ obtained from the product map $U(\mathfrak{a})^{\otimes 2} \to U(\mathfrak{a})$ by the symmetrization map, where $\mathfrak{a}$ is a Lie algebra.

Thus δ_S is the propic version of the image under the symmetrization map of the co-Poisson map $\delta_{U(\mathfrak{a})} : U(\mathfrak{a}) \to U(\mathfrak{a})^{\otimes 2}$, where $\mathfrak{a}$ is a Lie bialgebra.

1.21. **The morphisms $m_\Pi \in \mathbf{\Pi}((S\underline{\boxtimes}S)^{\otimes 2}, S\underline{\boxtimes}S)$ and $\Delta_0 \in \mathbf{\Pi}(S\underline{\boxtimes}S, (S\underline{\boxtimes}S)^{\otimes 2})$.** We introduce the propic version m_Π of the product map $U(\mathfrak{g})^{\otimes 2} \to U(\mathfrak{g})$, where $\mathfrak{g}$ is the double of a Lie bialgebra $\mathfrak{a}$, transported via the isomorphism $U(\mathfrak{g}) \simeq S(\mathfrak{a}) \otimes S(\mathfrak{a}^*)$ induced by the symmetrizations and the product map $U(\mathfrak{a}) \otimes U(\mathfrak{a}^*) \to U(\mathfrak{g})$.

We construct a prop morphism $\mathrm{LA} \to (\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id})(\pi^{\mathrm{left}})$, taking $\mu \in \mathrm{LA}(\wedge^2, \mathbf{id})$ to the sum of $\mu \in \mathrm{LBA}(\wedge^2, \mathbf{id}) \simeq \pi(\wedge^2 \boxtimes \mathbf{1}, \mathbf{id} \boxtimes \mathbf{1})$, $\delta \in \mathrm{LBA}(\mathbf{id}, \wedge^2) \subset \mathrm{LBA}(\mathbf{id}, T_2) \simeq \pi(\mathbf{id}\boxtimes\mathbf{id}, \mathbf{id}\boxtimes\mathbf{1})$, $\mu \in \mathrm{LBA}(\wedge^2, \mathbf{id}) \subset \mathrm{LBA}(T_2, \mathbf{id}) \simeq \pi(\mathbf{id} \boxtimes \mathbf{id}, \mathbf{1} \boxtimes \mathbf{id})$, and $\delta \in \mathrm{LBA}(\mathbf{id}, \wedge^2) \simeq \pi(\mathbf{1}\underline{\boxtimes}\wedge^2, \mathbf{1} \boxtimes \mathbf{id})$. This image is a morphism

$$\boldsymbol{\mu} : \wedge^2(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \simeq (\wedge^2 \boxtimes \mathbf{1}) \oplus (\mathbf{id} \boxtimes \mathbf{id}) \oplus (\mathbf{1} \boxtimes \wedge^2) \to (\mathbf{id} \boxtimes \mathbf{1}) \oplus (\mathbf{1} \boxtimes \mathbf{id})$$

of π^{left}. It has Π-degree 0.

Let us denote by

$$m_\pi \in (\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id})(\boldsymbol{\pi}^{\mathrm{left}})(S^{\otimes 2}, S) \simeq \boldsymbol{\pi}^{\mathrm{left}}(S^{\otimes 2}(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}), S\circ(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id})) \simeq \boldsymbol{\pi}^{\mathrm{left}}((S\boxtimes S)^{\otimes 2}, S\boxtimes S)$$

the image of $m_{\mathrm{PBW}} \in \mathbf{LA}(S^{\otimes 2}, S)$.

We denote by $m_\Pi \in \mathbf{\Pi}^{\mathrm{left}}((S\underline{\boxtimes}S)^{\boxtimes 2}, S\underline{\boxtimes}S) \subset \mathbf{\Pi}((S\underline{\boxtimes}S)^{\boxtimes 2}, S\underline{\boxtimes}S)$ the image of m_π. Then m_Π has Π-degree 0.

Then m_π is associative, therefore

$$m_\Pi \circ (m_\Pi \boxtimes \mathrm{id}_{S\underline{\boxtimes}S}) = m_\Pi \circ (\mathrm{id}_{S\underline{\boxtimes}S} \boxtimes m_\Pi). \tag{15}$$

We denote by $m_\Pi^{(2)} \in \mathbf{\Pi}((S\underline{\boxtimes}S)^{\otimes 3}, S\underline{\boxtimes}S) = \mathbf{\Pi}(S^{\boxtimes 3}\underline{\boxtimes}S^{\boxtimes 3}, S\underline{\boxtimes}S)$ the common value of both sides, and more generally by $m_\Pi^{(n)}$ the $n$fold iterate of m_Π.

Let us define $m_{ba} \in \boldsymbol{\pi}^{\mathrm{left}}((\mathbf{1} \boxtimes S) \otimes (S \boxtimes \mathbf{1}), S \boxtimes S)$ as $m_\pi \circ ((\mathrm{inj}_0 \boxtimes \mathrm{id}_S) \otimes (\mathrm{id}_S \boxtimes \mathrm{inj}_0))$ (where $\mathrm{inj}_0 : \mathbf{1} \to S$ is the canonical morphism).

Since $m_\pi \circ \mathrm{can}_{S\boxtimes S} = \mathrm{id}_{S\boxtimes S}$, where $\mathrm{can}_{S\boxtimes S} \in \mathbf{Sch}(S \boxtimes S, (S \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes S))$ is the canonical map, and since we have commutative diagrams

$$\begin{array}{ccc} (S \boxtimes \mathbf{1})^{\otimes 2} & \to & (S \boxtimes S)^{\otimes 2} \\ {\scriptstyle m_{\mathrm{PBW}}\boxtimes \mathrm{id}_{\mathbf{1}}}\downarrow & & \downarrow{\scriptstyle m_\pi} \\ S \boxtimes \mathbf{1} & \to & S \boxtimes S \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathbf{1} \boxtimes S)^{\otimes 2} & \to & (S \boxtimes S)^{\otimes 2} \\ {\scriptstyle \mathrm{id}_{\mathbf{1}} \boxtimes m^*_{\mathrm{PBW}}}\downarrow & & \downarrow{\scriptstyle m_\pi} \\ \mathbf{1} \boxtimes S & \to & S \boxtimes S \end{array} \tag{16}$$

we have

$$m_\Pi = (\tilde m_{\mathrm{PBW}} \boxtimes \tilde m^*_{\mathrm{PBW}}) \circ (\mathrm{id}_{S\underline{\boxtimes}\mathbf{1}} \boxtimes \tilde m_{ba} \boxtimes \mathrm{id}_{\mathbf{1}\underline{\boxtimes}S}) \circ \mathrm{can}^{\boxtimes 2}_{S\underline{\boxtimes}S},$$

where $\mathrm{can}_{S\underline{\boxtimes}S} \in \mathbf{\Pi}(S\underline{\boxtimes}S, (S\underline{\boxtimes}\mathbf{1}) \boxtimes (\mathbf{1}\underline{\boxtimes}S))$ is the canonical morphism, $\tilde m_{ba} \in \mathbf{\Pi}((\mathbf{1}\underline{\boxtimes}S) \boxtimes (S\underline{\boxtimes}\mathbf{1}), (S\underline{\boxtimes}\mathbf{1}) \boxtimes (\mathbf{1}\underline{\boxtimes}S))$ is the morphism derived from m_{ba}, $\tilde m_{\mathrm{PBW}} \in \mathbf{\Pi}((S\underline{\boxtimes}\mathbf{1})^{\boxtimes 2}, S\underline{\boxtimes}\mathbf{1})$ and $\tilde m^*_{\mathrm{PBW}} \in \mathbf{\Pi}((\mathbf{1}\underline{\boxtimes}S)^{\boxtimes 2}, \mathbf{1}\underline{\boxtimes}S)$ are the morphisms derived from m_{PBW} and m^*_{PBW}. It follows that a graph[5] for m_Π is as follows: set $F_1 = ... = G_2 = S$, so that it belongs to $\mathbf{\Pi}((F_1 \boxtimes F_2)\underline{\boxtimes}(G_1 \boxtimes G_2), F'\underline{\boxtimes}G')$, then the edges of the graph are $F_i \to F'$, $G' \to G_i$, $F_2 \to G_1$.

It follows that if we view $m_\Pi^{(n-1)}$ as an element of $\mathbf{\Pi}((\boxtimes_{i=1}^n F_i)\underline{\boxtimes}(\boxtimes_{i=1}^n G_i), F'\underline{\boxtimes}G')$, where $F_1 = ... = G' = S$, a graph for $m_\Pi^{(n-1)}$ is $F_i \to F'$, $G' \to G_i$ $(i = 1, ..., n)$, $F_j \to G_i$ $(1 \le i < j \le n)$.

We observe for later use that the morphism

$$T(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \to S \boxtimes S$$

[5] For $x \in \Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')$, $F = \boxtimes_{i\in I} F_i, ..., G' = \boxtimes_{j'\in J'} G'_{j'}$, and $S \subset (I \sqcup J') \times (I' \sqcup J)$, we say that x admits the graph S if $x \in \oplus_{S' \subset S} \Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')_{S'}$.

in $\boldsymbol{\pi}^{\text{left}}$, given by the direct sum over $n \geq 0$ of all compositions $(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id})^{\otimes n} \to (S \boxtimes S)^{\otimes n} \overset{m_\pi^{(n-1)}}{\to} S \boxtimes S$, is the cokernel of the morphism

$$T(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \otimes T_2(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \otimes T(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \to T(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id})$$

given by $m_T^{(2)} \circ \Big(\mathrm{id}_T \otimes \big((12) - (21) - \bar{\boldsymbol{\mu}}\big) \otimes \mathrm{id}_T \Big)$, where $\bar{\boldsymbol{\mu}}$ is the composed morphism

$$\bar{\boldsymbol{\mu}} : T_2(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \to \wedge^2(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}) \simeq \wedge^2(\mathbf{id}) \boxtimes \mathbf{1} \oplus \mathbf{id} \boxtimes \mathbf{id} \oplus \mathbf{1} \boxtimes \wedge^2(\mathbf{id})$$
$$\overset{\boldsymbol{\mu}}{\to} \mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id} = T_1(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}).$$

Let us define $\Delta_0 \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\boxtimes 2})$. Recall that $\Delta_0^S \in \mathbf{Sch}(S, S^{\otimes 2}) \subset \mathbf{LBA}(S, S^{\otimes 2}) \subset \mathbf{\Pi}(S \underline{\boxtimes} \mathbf{1}, (S \boxtimes S) \underline{\boxtimes} \mathbf{1})$ and let $m_0^S \in \mathbf{Sch}(S^{\otimes 2}, S) \subset \mathbf{LBA}(S^{\otimes 2}, S) = \mathbf{\Pi}(\mathbf{1} \underline{\boxtimes} S, \mathbf{1} \underline{\boxtimes} (S \boxtimes S))$ be the propic version of the product of the symmetric algebra $S(V)$. Set

$$\Delta_0 := \Delta_0^S \boxtimes m_0^S \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \boxtimes S) \underline{\boxtimes} (S \boxtimes S)) \simeq \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\boxtimes 2}).$$

A graph for this element is as follows. Set $F = G = ... = G_2' = S$, then $\Delta_0 \in \mathbf{\Pi}(F \underline{\boxtimes} G, (F_1' \boxtimes F_2') \underline{\boxtimes} (G_1' \boxtimes G_2'))$, and a graph is $F \to F_1'$, $F \to F_2'$, $G_1' \to G$, $G_2' \to G$.

Then both sides of the following equality are defined, and the equality holds:

$$\Delta_0 \circ m_\Pi = (m_\Pi \boxtimes m_\Pi) \circ (1324) \circ (\Delta_0 \boxtimes \Delta_0). \tag{17}$$

This follows from (14).

We also have commutative diagrams

$$\begin{array}{ccc} S \underline{\boxtimes} \mathbf{1} & \to & S \underline{\boxtimes} S \\ {\scriptstyle \Delta_0^S \boxtimes 1} \downarrow & & \downarrow {\scriptstyle \Delta_0} \\ (S \underline{\boxtimes} \mathbf{1})^{\otimes 2} & \to & (S \underline{\boxtimes} S)^{\otimes 2} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{1} \underline{\boxtimes} S & \to & S \underline{\boxtimes} S \\ {\scriptstyle 1 \boxtimes (\Delta_0^S)^t} \downarrow & & \downarrow {\scriptstyle \Delta_0} \\ (\mathbf{1} \underline{\boxtimes} S)^{\otimes 2} & \to & (S \underline{\boxtimes} S)^{\otimes 2} \end{array} \tag{18}$$

1.22. **The element $\delta_{S \underline{\boxtimes} S} \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\otimes 2})$.** We have $\mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\otimes 2}) \simeq \boldsymbol{\pi}(S \boxtimes S, S^{\otimes 2} \boxtimes S^{\otimes 2})$. Define

$$\delta_{S \boxtimes S} := \delta_S \boxtimes m_0^S + \Delta_0^S \boxtimes \delta_S^t \tag{19}$$

and $\delta_{S \underline{\boxtimes} S}$ as the image of this element in $\mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\otimes 2})$. Then $((12) + (21)) \circ \delta_{S \underline{\boxtimes} S} = 0$.

Let $r \in \Pi(\mathbf{1} \underline{\boxtimes} \mathbf{1}, (S \underline{\boxtimes} S)^{\boxtimes 2})$ be the image of the composition $S \to \mathbf{id} \to S$ in $\mathbf{Sch}(S, S) \subset \mathbf{LBA}(S, S) \simeq \boldsymbol{\pi}(\mathbf{1} \boxtimes \mathbf{1}, (S \otimes \mathbf{1}) \boxtimes (\mathbf{1} \otimes S)) \subset \boldsymbol{\pi}(\mathbf{1} \boxtimes \mathbf{1}, S^{\otimes 2} \boxtimes S^{\otimes 2}) \simeq \mathbf{\Pi}(\mathbf{1} \underline{\boxtimes} \mathbf{1}, S^{\boxtimes 2} \underline{\boxtimes} S^{\boxtimes 2})$. Then one checks that

$$\delta_{S \underline{\boxtimes} S} = m_\Pi^{\boxtimes 2} \circ (r \boxtimes \Delta_0 - \Delta_0 \boxtimes r),$$

which is a propic version of the statement that the co-Poisson structure on $U(D(\mathfrak{a}))$ is quasi-triangular, with r-matrix $r_{\mathfrak{a}}$. (19) also implies that the diagrams

$$\begin{array}{ccc} S \underline{\boxtimes} \mathbf{1} & \to & S \underline{\boxtimes} S \\ {\scriptstyle \delta_S \boxtimes 1} \downarrow & & \downarrow {\scriptstyle \delta_{S \underline{\boxtimes} S}} \\ (S \underline{\boxtimes} \mathbf{1})^{\otimes 2} & \to & (S \underline{\boxtimes} S)^{\otimes 2} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{1} \underline{\boxtimes} S & \to & S \underline{\boxtimes} S \\ {\scriptstyle 1 \boxtimes \delta_S^t} \downarrow & & \downarrow {\scriptstyle \delta_{S \underline{\boxtimes} S}} \\ (\mathbf{1} \underline{\boxtimes} S)^{\otimes 2} & \to & (S \underline{\boxtimes} S)^{\otimes 2} \end{array}$$

commute.

1.23. **The morphism $\Xi_f \in \mathbf{\Pi}_f(S \underline{\boxtimes} S, S \underline{\boxtimes} S)^\times$.** If $\mathfrak{a}$ is a Lie bialgebra and $f \in \wedge^2(\mathfrak{a})$ is a twist (we denote by $\mathfrak{a}_f$ the Lie bialgebra $(\mathfrak{a}, \delta + \mathrm{ad}(f))$), then the doubles $D(\mathfrak{a}) \simeq \mathfrak{a} \oplus \mathfrak{a}^*$ and $D(\mathfrak{a}_f) \simeq \mathfrak{a} \oplus \mathfrak{a}^*$ are Lie algebra isomorphic, the isomorphism $D(\mathfrak{a}) \to D(\mathfrak{a}_f)$ being given by the automorphism of $\mathfrak{a} \oplus \mathfrak{a}^*$, $(a, 0) \mapsto (a, 0)$ and $(0, \alpha) \mapsto ((\mathrm{id}_{\mathfrak{a}} \otimes \alpha)(f), \alpha)$. The composed isomorphism $S(\mathfrak{a}) \otimes S(\mathfrak{a}^*) \to U(D(\mathfrak{a})) \simeq U(D(\mathfrak{a}_f)) \simeq S(\mathfrak{a}) \otimes S(\mathfrak{a}^*)$ has a propic version $\Xi_f \in \mathbf{\Pi}_f(S \underline{\boxtimes} S, S \underline{\boxtimes} S)^\times$, which we now construct.

We define

$$\iota \in \pi_f^{\text{left}}(\mathbf{id} \boxtimes \mathbf{1}, \mathbf{id} \boxtimes \mathbf{1}) \oplus \pi_f^{\text{left}}(\mathbf{1} \boxtimes \mathbf{id}, \mathbf{id} \boxtimes \mathbf{1}) \oplus \pi_f^{\text{left}}(\mathbf{1} \boxtimes \mathbf{id}, \mathbf{1} \boxtimes \mathbf{id}) \subset \pi_f^{\text{left}}(\mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id}, \mathbf{id} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \mathbf{id})$$

as the sum of $\pi_f(\mathrm{id}_{\mathbf{id}\boxtimes\mathbf{1}})$, of the element corresponding to f, and of $\pi_f(\mathrm{id}_{\mathbf{1}\boxtimes\mathbf{id}})$. Then ι is homogeneous of degree 0 (in the case of the middle element, the degree of f is compensated by the fact that the source and the target have different degrees).

Then the diagram

$$\begin{array}{ccc} (T\otimes T_2\otimes T)(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T(\iota)\otimes T_2(\iota)\otimes T(\iota)}{\to} & (T\otimes T_2\otimes T)(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \\ {\scriptstyle \kappa_1^\pi\left(m_T^{(2)}\circ(\mathrm{id}_T\otimes((12)-(21)-\bar\mu)\otimes\mathrm{id}_T)\right)}\downarrow & & \downarrow{\scriptstyle \kappa_2^\pi\left(m_T^{(2)}\circ(\mathrm{id}_T\otimes((12)-(21)-\bar\mu)\otimes\mathrm{id}_T)\right)} \\ T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T(\iota)}{\to} & T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \end{array}$$

commutes in $\boldsymbol{\pi}_f^{\mathrm{left}}$. Taking cokernels, we get a morphism $\xi_f\in\boldsymbol{\pi}_f^{\mathrm{left}}(S\boxtimes S,S\boxtimes S)$. We denote by Ξ_f its image in $\mathbf{\Pi}_f(S\underline{\boxtimes}S,S\underline{\boxtimes}S)$.

A lift of Ξ_f to $\mathbf{\Pi}\Big(((S\circ\wedge^2)\boxtimes S)\underline{\boxtimes}S,S\underline{\boxtimes}S\Big)$ (which we write $\mathbf{\Pi}\Big(((S\circ\wedge^2)\boxtimes F)\underline{\boxtimes}G,F'\underline{\boxtimes}G'\Big)$) admits the graph $F\to F'$, $G'\to G$, $F'\to G'$, $S\circ\wedge^2\to G$, $S\circ\wedge^2\to F'$.

Since ι is invertible, so is Ξ_f.

1.24. **Relations between κ_i^Π, Ξ_f and m_Π, Δ_0.** Let us now study the relations of Ξ_f with m_Π. In the case of a Lie bialgebra with twist $(\mathfrak{a},f)$, since $D(\mathfrak{a})\to D(\mathfrak{a}_f)$ is a Lie algebra isomorphism, the diagram

$$\begin{array}{ccc} (S(\mathfrak{a})\otimes S(\mathfrak{a}^*))^{\otimes 2} & \to & S(\mathfrak{a})\otimes S(\mathfrak{a}^*) \\ \downarrow & & \downarrow \\ (S(\mathfrak{a})\otimes S(\mathfrak{a}^*))^{\otimes 2} & \to & S(\mathfrak{a})\otimes S(\mathfrak{a}^*) \end{array}$$

commutes, where the upper (resp., lower) arrow is induced by the product in $U(D(\mathfrak{a}))$ (resp., $U(D(\mathfrak{a}_f))$) and the vertical arrows are given by the above automorphism of $S(\mathfrak{a})\otimes S(\mathfrak{a}^*)$. The propic version of this statement is that both terms of the following identity are defined, and the identity holds:

$$\kappa_2^\Pi(m_\Pi)=\Xi_f\circ\kappa_1^\Pi(m_\Pi)\circ(\Xi_f^{-1})^{\boxtimes 2}. \tag{20}$$

The proof of this statement relies on the properties of a cokernel and on the commutativity of the diagram

$$\begin{array}{ccc} T^{\otimes 2}(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T^{\otimes 2}(\iota)}{\to} & T^{\otimes 2}(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \\ {\scriptstyle \kappa_1^\pi(m_T^{(2)})}\downarrow & & \downarrow{\scriptstyle \kappa_2^\pi(m_T^{(2)})} \\ T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T(\iota)}{\to} & T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \end{array}$$

We now study the relation of Ξ_f with Δ_0. In the case of a Lie bialgebra with twist $(\mathfrak{a},f)$, the isomorphism $U(D(\mathfrak{a}))\to U(D(\mathfrak{a}_f))$ is also compatible with the (cocommutative) bialgebra structures, as it is induced by a Lie algebra isomorphism. The propic version of this statement is that both sides of the following identity are defined, as the identity holds

$$\kappa_2^\Pi(\Delta_0)=\Xi_f^{\boxtimes 2}\circ\kappa_1^\Pi(\Delta_0)\circ\Xi_f^{-1}. \tag{21}$$

This identity follows from the commutativity of

$$\begin{array}{ccc} T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T(\iota)}{\to} & T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \\ {\scriptstyle \pi(\Delta_T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}))}\downarrow & & \downarrow{\scriptstyle \pi(\Delta_T(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}))} \\ T^{\otimes 2}(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{T^{\otimes 2}(\iota)}{\to} & T^{\otimes 2}(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) \end{array}$$

1.25. **Relations between τ_Π, m_Π and Δ_0.** Define $\omega_S \in \mathbf{Sch}(S,S)^\times$ as $\widehat{\oplus}_{n\geq 0}(-1)^n \mathrm{id}_{S^n}$.

Lemma 1.19. *We have*

$$\tau_\Pi(m_\Pi) = (\mathrm{id}_S \underline{\boxtimes} \omega_S)\circ m_\Pi \circ ((\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\otimes 2})^{-1}, \quad \tau_\Pi(\Delta_0) = (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\otimes 2}\circ \Delta_0 \circ (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{-1}.$$

Proof. The first statement follows the the commutativity of the diagram

$$\begin{array}{ccc} \wedge^2(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{\boldsymbol{\mu}}{\to} & \mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id} \\ {\scriptstyle \wedge^2(\mathrm{id}_{\mathbf{id}\boxtimes\mathbf{1}}\oplus(-\mathrm{id}_{\mathbf{1}\boxtimes\mathbf{id}}))}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbf{id}\boxtimes\mathbf{1}}\oplus(-\mathrm{id}_{\mathbf{1}\boxtimes\mathbf{id}})} \\ \wedge^2(\mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id}) & \stackrel{\tau_\pi(\boldsymbol{\mu})}{\to} & \mathbf{id}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\mathbf{id} \end{array}$$

The second statement follows from $m_0^S = \omega_S \circ m_0^S \circ (\omega_S^{\boxtimes 2})^{-1}$. □

Remark 1.20. Lemma 1.19 is the propic version of the following statement. Let $\mathfrak{a}$ be a Lie bialgebra and let $\mathfrak{a}' := \mathfrak{a}^{\mathrm{cop}}$ be $\mathfrak{a}$ with opposite coproduct. Its double $\mathfrak{g}'$ is Lie algebra isomorphic to $\mathfrak{g}$, using the automorphism $\mathrm{id}_{\mathfrak{a}} \oplus(-\mathrm{id}_{\mathfrak{a}^*})$ of $\mathfrak{a}\oplus\mathfrak{a}^*$. The bialgebras $U(\mathfrak{g})$ and $U(\mathfrak{g}')$ are therefore isomorphic, the isomorphism being given by $U(\mathfrak{g}) \simeq S(\mathfrak{a}\oplus\mathfrak{a}^*) \stackrel{S(\mathrm{id}_{\mathfrak{a}}\oplus(-\mathrm{id}_{\mathfrak{a}^*}))}{\to} S(\mathfrak{a}\oplus\mathfrak{a}^*) \simeq U(\mathfrak{g}')$.

2. The $\mathcal{X}$-algebras $\mathbf{U}_n$ and $\mathbf{U}_{n,f}$

2.1. **The category $\mathcal{X}$.** Let $\mathcal{X}$ be the category where objects are finite sets and morphisms are partially defined functions. A $\mathcal{X}$-vector space (resp., algebra) is a contravariant functor $\mathcal{X} \to \mathrm{Vect}$ (resp., $\mathcal{X} \to \mathrm{Alg}$, where Alg is the category of algebras). A $\mathcal{X}$-vector space (resp., algebra) is the same as a collection $(V_s)_{s\geq 0}$ of vector spaces (resp., algebras), together with a collection of morphisms (called insertion-coproduct morphisms) $V_s \to V_t$, $x \mapsto x^\phi$, for each function $\phi : [t] \to [s]$, satisfying the chain rule. Instead of x^ϕ, we often write $x^{\phi^{-1}(1),\ldots,\phi^{-1}(s)}$. An example of a $\mathcal{X}$-vector space (resp., $\mathcal{X}$-algebra) is $V_s = H^{\otimes s}$, where H is a cocommutative coalgebra (resp., bialgebra). Then x^ϕ is obtained from x by applying the $(|\phi^{-1}(\alpha)|-1)$th iterated coproduct to the component α of x and plugging the result in the factors $\phi^{-1}(\alpha)$, for $\alpha = 1, ..., s$.

2.2. **The $\mathcal{X}$-algebra $\mathbf{U}_n$.** Let us set $\mathbf{U}_n := \Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (S\underline{\boxtimes}S)^{\boxtimes n})$. Let us equip it with the Π-degree: the Π-degree of a homogeneous element of $\Pi(F\underline{\boxtimes}G, F'\underline{\boxtimes}G')$ is $\geq -|F|-|G|$, therefore $\mathbf{U}_n$ is $\mathbb{N}$-graded.

For $x, y \in \mathbf{U}_n$, the composition $m_\Pi^{\boxtimes n}\circ(1, n+1, 2, n+2, ...)\circ(x\boxtimes y)$ is well-defined. We set

$$xy := m_\Pi^{\boxtimes n}\circ(1, n+1, 2, n+2, ...)\circ(x\boxtimes y) \in \mathbf{U}_n.$$

It follows from (15) and from $\deg_\Pi(m_\Pi) = 0$ that the map $x\otimes y \mapsto xy$ defines an associative product of degree 0 on $\mathbf{U}_n$.

Let $\mathrm{Coalg}_{\mathrm{coco}}$ be the prop of cocommutative bialgebras, let $\phi : [m] \to [n]$ and let $\Delta^\phi \in \mathrm{Coalg}_{\mathrm{coco}}(T_n, T_m)$ be the corresponding element. We have a prop morphism $\mathrm{Coalg}_{\mathrm{coco}} \to (S\underline{\boxtimes}S)(\Pi^{\mathrm{right}})$, induced by (coproduct) $\mapsto \Delta_0$. We denote by Δ_0^ϕ the image of Δ^ϕ by this morphism, and we set

$$x^\phi := \Delta_0^\phi \circ x.$$

It then follows from (17) that the family of all $(\mathbf{U}_n)_{n\geq 0}$, equipped with these insertion-coproduct morphisms, is a $\mathcal{C}$-algebra.

Remark 2.1. The element r defined in Subsection 1.22 belongs to $\mathbf{U}_2$; one checks that it satisfies the classical Yang-Baxter equation $[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0$ in $\mathbf{U}_3$. □

Lemma 2.2. *The map $\mathbf{U}_n \to \mathbf{U}_n$, $x \mapsto (\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes n} \circ \tau_\Pi(x)$ is a automorphism of $\mathbb{N}$-graded $\mathcal{X}$-algebra.*

Proof. This follows from Lemma 1.19 and from the fact that τ_Π has degree 0. □

Remark 2.3. One checks that $(\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes 2} \circ \tau_\Pi(r) = -r$, so the above automorphism will be denoted $x \mapsto x(-r)$. It is involutive, i.e., $(x(-r))(-r) = x$.

2.3. **The $\mathcal{X}$-algebra $\mathbf{U}_{n,f}$.** We set $\mathbf{U}_{n,f} := \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (S\underline{\boxtimes}S)^{\boxtimes n})$. The Π_f-degree induces a grading on $\mathbf{U}_{n,f}$.

Lemma 2.4. *$\mathbf{U}_{n,f}$ is a $\mathbb{N}$-graded vector space.*

Proof. Let x be homogeneous in the image of $\mathrm{LBA}((S^k \circ \wedge^2) \otimes F \otimes G', F' \otimes G) \to \mathrm{LBA}_f(F \otimes G', F' \otimes G)$. Let us show that $\deg_{\mathrm{LBA}_f}(x) \geq k - |F| - |G'|$. Indeed, if x belongs to

$$\mathrm{LCA}(S^k \circ \wedge^2, W_1 \otimes W_1) \otimes \mathrm{LCA}(F, Z_1 \otimes Z_2) \otimes \mathrm{LCA}(G', Z_3 \otimes Z_4) \otimes \mathrm{LA}(W_1 \otimes Z_1 \otimes Z_3, F) \otimes \mathrm{LA}(W_2 \otimes Z_2 \otimes Z_4, G),$$

its degree is $\deg_{\mathrm{LBA}_f}(x) = k + \deg_\delta(x) = k + |W_1| + |W_2| + |Z_1| + ... + |Z_4| - |F| - |G'| - 2k = (|W_1| + |W_2| - 2k) + |Z_1| + ... + |Z_4| - |F| - |G'| + k$. Now $|W_1| + |W_2| \geq 2k$, so $\deg_{\mathrm{LBA}_f}(x) \geq k - |F| - |G'|$. Therefore $\deg_{\Pi_f}(x) \geq k - |F| - |G|$. In our case, $F = G = \mathbf{1}$, so $\deg_{\mathrm{LBA}_f}(x) \geq 0$. □

For $x, y \in \mathbf{U}_{n,f}$, we define xy and x^ϕ as above, replacing m_Π and Δ_0^ϕ by $\kappa_1^\Pi(m_\Pi)$ and $\kappa_1^\Pi(\Delta_0^\phi)$. This makes $\mathbf{U}_{n,f}$ into a $\mathbb{N}$-graded $\mathcal{X}$-algebra.

Lemma 2.5. *The maps $\mathbf{U}_n \to \mathbf{U}_{n,f}$, $x \mapsto \kappa_1^\Pi(x)$ and $x \mapsto (\Xi_f^{-1})^{\boxtimes n} \circ \kappa_2^\Pi(x)$ are morphisms of $\mathbb{N}$-graded $\mathcal{C}$-algebras.*

Proof. This follows from (20), (21) and the fact that $\kappa_i^\Pi(m_\Pi)$, $\kappa_i^\Pi(\Delta_0)$ have degree 0. □

Remark 2.6. Let f be the image of $(\mathrm{inj}_1^{\otimes 2} \circ \mathrm{can}) \otimes \mathrm{pr}_0^{\otimes 2} \in \mathrm{LBA}(\wedge^2 \otimes S^{\otimes 2}, S^{\otimes 2}) \subset \mathrm{LBA}((S \circ \wedge^2) \otimes S^{\otimes 2}, S^{\otimes 2}) \to \mathrm{LBA}_f(S^{\otimes 2}, S^{\otimes 2}) \simeq \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (S\underline{\boxtimes}S)^{\otimes 2}) = \mathbf{U}_{n,f}$, where $\mathrm{can} : \wedge^2 \to \mathbf{id}^{\otimes 2}$ is the canonical morphism. We then have $(\Xi_f^{-1})^{\boxtimes 2} \circ \kappa_2^\Pi(r) = \kappa_1^\Pi(r) + f$. The morphisms $\mathbf{U}_n \to \mathbf{U}_{n,f}$, $x \mapsto \kappa_1^\Pi(x)$ and $x \mapsto (\Xi_f^{-1})^{\boxtimes 2} \circ \kappa_2^\Pi(x)$ will be denoted $x \mapsto x(r)$ and $x \mapsto x(r+f)$.

2.4. **The algebras $\mathbf{U}_{n,f}^{c_1 \ldots c_n}$.** For $c_1, ..., c_n \in \{a, b\}$, we set $\mathbf{U}_{n,f}^{c_1 \ldots c_n} := \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, F_{c_1} \boxtimes ... \boxtimes F_{c_n})$, where $F_a = S\underline{\boxtimes}\mathbf{1}$ and $F_b = \mathbf{1}\underline{\boxtimes}S$. Then $\mathbf{U}_{n,f}^{c_1 \ldots c_n} \subset \mathbf{U}_{n,f}$ is a graded subspace. The diagrams (16) imply that it is also a subalgebra.

Diagrams (18) also imply that for $\phi : [m] \to [n]$ partially defined, Δ^ϕ takes $\mathbf{U}_{n,f}^{c_1 \ldots c_n}$ to $\mathbf{U}_{m,f}^{c'_1 \ldots c'_m}$, where $c'_1, ..., c'_m$ are such that $c'_k = c_{\phi(k)}$ for any k in the domain of ϕ.

In particular, $(\mathbf{U}_{n,f}^{a \ldots a})_{n \geq 0}$ is a $\mathbb{N}$-graded $\mathcal{X}$-algebra.

2.5. **Hochschild cohomology of $\mathbf{U}_{n,f}$ and $\mathbf{U}_{n,f}^{a \ldots a}$.** The co-Hochschild complex of a $\mathcal{X}$-vector space $(V_n)_{n \geq 0}$ is given by the differentials $d^1 : V_1 \to V_2$, $x \mapsto x^{12} - x^1 - x^2$, $d^2 : V_2 \to V_3$, $x \mapsto x^{12,3} - x^{1,23} - x^{2,3} + x^{1,2}$, etc. We denote the corresponding cohomology groups by $H^n(V_*)$.

If $B = (B_{\rho,\sigma}) \in \mathrm{Ob}(\mathbf{Sch}_2)$ and $C \in \mathrm{Sch}$, we set $B_C := \oplus_{\rho,\sigma} B_{\rho,\sigma} \otimes \mathrm{LBA}(C \otimes Z_\sigma, Z_\rho)$. If $\alpha : C \to D$ is a morphism in LBA, then we set $B_\alpha := \mathrm{Coker}(B_D \to B_C) = \oplus_{\rho,\sigma} B_{\rho,\sigma} \otimes \mathrm{LBA}_\alpha(Z_\sigma, Z_\rho)$. In the case of the above morphism $\alpha : \wedge^3 \otimes (S \circ \wedge^2) \to S \circ \wedge^2$, we define in this way spaces B_f.

In particular, for $F, G \in \mathrm{Ob}(\mathbf{Sch}_{(1)})$, we have isomorphisms $\Pi(\mathbf{1}\underline{\boxtimes}\mathbf{1}, F\underline{\boxtimes}G) \simeq (c(F) \boxtimes c(G))_{\mathbf{1}}$ and $\Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, F\underline{\boxtimes}G) \simeq (c(F) \boxtimes c(G))_f$.

Lemma 2.7. *1) $H^n(\mathbf{U}_{*,f}^{a\dots a}) \simeq (\wedge^n \boxtimes \mathbf{1})_f$. If we set $C_a^n := \mathrm{Ker}(d : \mathbf{U}_{n,f}^{a\dots a} \to \mathbf{U}_{n+1,f}^{a\dots a})$, then* $\mathrm{Alt} = (n!)^{-1}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\sigma : \mathbf{U}_{n,f}^{a\dots a} \to \mathbf{U}_{n,f}^{a\dots a}$ *restricts to a map $C_a^n \to (\wedge^n \boxtimes \mathbf{1})_f$, which factors through the above isomorphism.*

2) $H^n(\mathbf{U}_{*,f}) \simeq (\Delta(\wedge^n))_f = \oplus_{p,q|p+q=n}(\wedge^p\boxtimes\wedge^q)_f$. If we set $C_a^n := \mathrm{Ker}(d : \mathbf{U}_{n,f}^{a\dots a} \to \mathbf{U}_{n+1,f}^{a\dots a})$, then $\mathrm{Alt} : \mathbf{U}_{n,f} \to \mathbf{U}_{n,f}$ *restricts to a map $C^n \to (\Delta(\wedge^n))_f$, which factors through the above isomorphism.*

Proof. 1) We have a co-Hochschild complex $S \to S^{\otimes 2} \to S^{\otimes 3} \to \dots$ in $\mathbf{Sch}$. It is defined as above, where $(x \mapsto x^{12,3})$ is replaced by the element of $\Delta_0^S \boxtimes \mathrm{id}_S \in \mathbf{Sch}(S^{\otimes 2}, S^{\otimes 3})$, etc. We express it as the sum of an acyclic complex $\Sigma_1 \to \Sigma_2 \to \dots$ and a complex with zero differential $\wedge^1 \to \wedge^2 \to \dots$

The inclusion $\wedge^n \to S^{\otimes n}$ is given by the composition $\wedge^n \subset \mathbf{id}^{\otimes n} \subset S^{\otimes n}$.

The inclusion $\Sigma_n \subset S^{\otimes n}$ is defined as follows: $\Sigma_n := \oplus_{k_1,\dots,k_n\neq 1}(\otimes_{i=1}^n S^{k_i}) \oplus \rho_n$, where $\rho_n \subset (S^1)^{\otimes n} = \mathbf{id}^{\otimes n}$ is the sum of all the images of the pairwise symmetrization maps $\mathrm{id}+(ji) : \mathbf{id}^{\otimes n} \to \mathbf{id}^{\otimes n}$, where $i < j \in [n]$. Then we have a direct sum decomposition $S^{\otimes n} = \Sigma_n \oplus \wedge^n$. One checks that this is a decomposition of complexes, where $\wedge^n$ has zero differential.

In particular, when V is a vector space, the co-Hochschild complex $V \to S^2(V) \to \dots$ decomposes as the sum of the complexes $\wedge^n(V)$ and $\Sigma_n(V)$. Since the cohomology is reduced to $\wedge^n(V)$, the complex $\Sigma_n(V)$ is acyclic. It has therefore a homotopy $\Sigma_n(V) \stackrel{K_n(V)}{\to} \Sigma_{n-1}(V)$, which has a propic version $\Sigma_n \stackrel{K_n}{\to} \Sigma_{n-1}$.

Recall that $\mathbf{U}_{n,f}^{a\dots a} \simeq (S^{\otimes n}\boxtimes\mathbf{1})_f$. The co-Hochschild complex for the latter space decomposes as the sum of $(\wedge^n \boxtimes \mathbf{1})_f$ with zero differential and $(\Sigma_n \boxtimes \mathbf{1})_f$, which admits a homotopy and is therefore acyclic. It follows that $H^n(\mathbf{U}_{*,f}^{a\dots a}) = (\wedge^n\boxtimes\mathbf{1})_f$. Then $C_a^n = (\wedge^n\boxtimes\mathbf{1})_f \oplus d((\Sigma_{n-1}\boxtimes\mathbf{1})_f)$. The restriction of Alt to C_a^n is then the projection on the first summand of this decomposition, which implies the second result. This proves 1).

Let us prove 2). We have $\mathbf{U}_{n,f} \simeq (S^{\otimes n}\boxtimes S^{\otimes n})_f \simeq (\Delta(S^{\otimes n}))_f$, where $\Delta : \mathrm{Ob}(\mathrm{Sch}) \to \mathrm{Ob}(\mathrm{Sch}_2)$ has been defined in Section 1.1.1.

We then have a decomposition $\mathbf{U}_{n,f} \simeq (\Delta(\wedge^n))_f \oplus (\Delta(\Sigma_n))_f$, where the first complex has zero differential and the second complex admits a homotopy and is therefore acyclic. Therefore $H^n(\mathbf{U}_{*,f}) = (\Delta(\wedge^n))_f$. As before, $C^n = (\Delta(\wedge^n))_f \oplus d(\Delta(\Sigma_{n-1})_f)$, and the restriction of Alt to C^n is the projection on the first summand. This proves 2). $\square$

Remark 2.8. One can prove that for $B \in \mathrm{Ob}(\mathbf{Sch_2})$, we have

$$B_{\mathbf{1}} = \oplus_{N\geq 0}\Big(B(\mathrm{Lie}(a_1,\dots,a_N) \oplus \mathrm{Lie}(b_1,\dots,b_N))_{\sum_{i=1}^N(\alpha_i+\beta_i)}\Big)_{\mathfrak{S}_N},$$

where $\mathrm{Lie}(x_1,\dots,x_N)$ is the free Lie algebra with generators $x_1,\dots,x_N$, the generators a_i, b_i have degrees $\alpha_i,\beta_i \in \oplus_{i=1}^N(\mathbb{N}\alpha_i\oplus\mathbb{N}\beta_i)$. Here the index $\sum_{i=1}^N(\alpha_i+\beta_i)$ means the part of degree $\sum_{i=1}^N(\alpha_i+\beta_i)$, and the index $\mathfrak{S}_N$ means the space of coinvariants w.r.t. the diagonal action of $\mathfrak{S}_N$ on generators a_i, b_i, $i = 1,\dots,N$.

Using the symmetrization map, we then get

$$\mathbf{U}_n \simeq \oplus_{N\geq 0}\Big(((\mathbf{k}\langle a_1,\dots,a_N\rangle\mathbf{k}\langle b_1,\dots,b_N\rangle)^{\otimes n})_{\sum_{i=1}^n(\alpha_i+\beta_i)}\Big)_{\mathfrak{S}_N},$$

where $\mathbf{k}\langle x_1,\dots,x_N\rangle$ is the free algebra with generators $x_1,\dots,x_N$, and $\mathbf{k}\langle a_1,\dots,a_N\rangle\mathbf{k}\langle b_1,\dots,b_N\rangle$ is the image of the product map $\mathbf{k}\langle a_1,\dots,a_N\rangle \otimes \mathbf{k}\langle b_1,\dots,b_N\rangle \to \mathbf{k}\langle a_1,\dots,b_N\rangle$. So $\mathbf{U}_n$ identifies with $(U(\mathfrak{g})^{\otimes n})_{\mathrm{univ}}$ (see [Enr1]).

Then $\mathbf{U}_n \subset \oplus_{N\geq 0}(\mathbf{k}\langle a_1,\dots,b_N\rangle^{\otimes n})_{\mathfrak{S}_N}$. This inclusion is compatible with the $\mathcal{X}$-structure on the right induced by the coalgebra structure of $\mathbf{k}\langle a_1,\dots,b_N\rangle$. The space $\mathbf{U}_n^{c_1,\dots,c_n}$ identifies with $\oplus_{N\geq 0}((F_{c_1}^N\otimes\dots\otimes F_{c_n}^N)_{\sum_{i=1}^N(\alpha_i+\beta_i)})_{\mathfrak{S}_N}$, where $F_a^N = \mathbf{k}\langle a_1,\dots,a_N\rangle$ and $F_b^N = \mathbf{k}\langle b_1,\dots,b_N\rangle$, and

if $(c_i, c'_i) \neq (b, a)$ for any i, then the product $\mathbf{U}_n^{c_1 \dots c_n} \otimes \mathbf{U}_n^{c'_1 \dots c'_n} \to \mathbf{U}_n$ is induced by the maps $F_c^N \otimes F_{c'}^M \to \mathbf{k}\langle a_1, ..., b_{N+M}\rangle$, $x(c_1, ..., c_N) \otimes x'(c'_1, ..., c'_M) \mapsto x(c_1, ..., c_N)x'(c'_{N+1}, ..., c'_{N+M})$.

3. Injectivity of a map

Let n, m be integers ≥ 0. Define $\mu_m \in \mathrm{LA}(T_m \otimes \mathbf{id}, T_m)$ as the propic version of the map $x_1 \otimes ... \otimes x_m \otimes x \mapsto \sum_{i=1}^m x_1 \otimes ... \otimes [x_i, x] \otimes ... \otimes x_m$. Define a linear map

$$i_{n,m} : \mathrm{LA}(T_n \otimes T_m, \mathbf{id}) \to \mathrm{LA}(T_n \otimes T_m \otimes \mathbf{id}, \mathbf{id}),$$
$$\lambda \mapsto \lambda \circ (\mathrm{id}_{T_n} \otimes \mu_m).$$

Define also a linear map

$$c_{n',n,m} : \mathrm{LA}(T_{n'}, T_n) \otimes \mathrm{LA}(T_n \otimes T_m, \mathbf{id}) \to \mathrm{LA}(T_{n'} \otimes T_m, \mathbf{id}),$$
$$\lambda_0 \otimes \lambda \mapsto \lambda \circ (\lambda_0 \boxtimes \mathrm{id}_{T_m}).$$

Then $i_{n,m}$ is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant, and the diagram

$$\begin{array}{ccc} \mathrm{LA}(T_{n'}, T_n) \otimes \mathrm{LA}(T_n \otimes T_m, \mathbf{id}) & \stackrel{\mathrm{id} \otimes i_{n,m}}{\to} & \mathrm{LA}(T_{n'}, T_n) \otimes \mathrm{LA}(T_n \otimes T_m \otimes \mathbf{id}, \mathbf{id}) \\ {\scriptstyle c_{n',n,m}}\downarrow & & \downarrow{\scriptstyle c_{n',n,m+1}} \\ \mathrm{LA}(T_{n'} \otimes T_m, \mathbf{id}) & \stackrel{i_{n',m}}{\to} & \mathrm{LA}(T_{n'} \otimes T_m \otimes \mathbf{id}, \mathbf{id}) \end{array}$$

commutes.

Lemma 3.1. *There exists a map $p_{n,m} : \mathrm{LA}(T_n \otimes T_m \otimes \mathbf{id}, \mathbf{id}) \to \mathrm{LA}(T_n \otimes T_m, \mathbf{id})$, such that $p_{n,m} \circ i_{n,m} = \mathrm{id}$, which is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant, and such that the diagram*

$$\begin{array}{ccc} \mathrm{LA}(T_{n'}, T_n) \otimes \mathrm{LA}(T_n \otimes T_m \otimes \mathbf{id}, \mathbf{id}) & \stackrel{\mathrm{id} \otimes p_{n,m}}{\to} & \mathrm{LA}(T_{n'}, T_n) \otimes \mathrm{LA}(T_n \otimes T_m, \mathbf{id}) \\ {\scriptstyle c_{n',n,m+1}}\downarrow & & \downarrow{\scriptstyle c_{n',n,m}} \\ \mathrm{LA}(T_{n'} \otimes T_m \otimes \mathbf{id}, \mathbf{id}) & \stackrel{p_{n',m}}{\to} & \mathrm{LA}(T_{n'} \otimes T_m, \mathbf{id}) \end{array}$$

commutes.

Proof. Let us first recall some results on free Lie algebras. Let $\mathrm{L}(x_1, ..., x_s)$ (resp., $\mathrm{A}(x_1, ..., x_s)$) the multilinear part of the free Lie (resp., associative) algebra generated by $x_1, ..., x_s$. Then $\mathrm{L}(x_1, ..., x_s) \subset \mathrm{A}(x_1, ..., x_s)$. For any $i = 1, ..., s$, we have an isomorphism $\mathrm{L}(x_1, ..., x_s) \stackrel{\sim}{\to} \mathrm{A}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_s)$, given by $\mathrm{L}(x_1, ..., x_s) \ni P(x_1, ..., x_s) \mapsto P_{x_i}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_s)$, where P_{x_i} is the element such that P decomposes as $P_{x_i} x_i$ + sum of terms not ending with x_i. The inverse isomorphism is given by $\mathrm{A}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_s) \ni Q \mapsto \mathrm{ad}(Q)(x_s)$, where $\mathrm{ad}(x_{i_1} ... x_{i_{s-1}})(x_i) = [x_{i_1}, [x_{i_2}, ..., [x_{i_{s-1}}, x_i]]]$.

Let us now prove the lemma. We have an isomorphism $\mathrm{LA}(T_n, \mathbf{id}) \simeq \mathrm{L}(x_1, ..., x_n)$. The map $i_{n,m}$ is given by

$$i_{n,m} : \mathrm{L}(x_1, ..., x_{n+m}) \to \mathrm{L}(x_1, ..., x_{n+m+1}),$$
$$P(x_1, ..., x_{n+m}) \mapsto \sum_{i=1}^m P(x_1, ..., [x_{n+i}, x_{n+m+1}], ..., x_{n+i}).$$

Define

$$p_{n,m} : \mathrm{L}(x_1, ..., x_{n+m+1}) \to \mathrm{L}(x_1, ..., x_{n+m}),$$
$$Q(x_1, ..., x_{n+m+1}) \mapsto \frac{1}{m} \sum_{i=1}^m \mathrm{ad}(Q_{x_{n+i} x_{n+m+1}})(x_{n+i}),$$

where $Q_{x_{n+i} x_{n+m+1}}$ is the element of $\mathrm{A}(x_1, ..., x_{n+i-1}, x_{n+i+1}, ..., x_{n+m})$ such that

$$Q = Q_{x_{n+i} x_{n+m+1}} x_{n+i} x_{n+m+1} \text{ + terms not ending by } x_{n+i} x_{n+m+1}.$$

Let us show that $p_{n,m} \circ i_{n,m} = \mathrm{id}$. If $Q_i := P(x_1, ..., [x_{n+i}, x_{n+m+1}], ..., x_{n+m})$, then $(Q_i)_{x_{n+j} x_{n+m+1}} = 0$ if $i \neq j$, and equals $P_{x_{n+j}}$ if $i = j$. So $(i_{n,m}(P))_{x_{n+j} x_{n+m+1}} = P_{x_{n+j}}$.

So $\mathrm{ad}\Big(\big(i_{n,m}(P)\big)_{x_{n+j}x_{x_{n+m+1}}}\Big)(x_{n+j}) = \mathrm{ad}(P_{x_{n+j}})(x_{n+j}) = P$. Averaging over $i \in [m]$, we get $p_{n,m}(i_{n,m}(P)) = P$.

Let us show that $p_{n,m}$ is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant. The $\mathfrak{S}_n$-equivariance is clear. Let us show the $\mathfrak{S}_m$-equivariance. Let $\tau \in \mathfrak{S}_m$. We have $Q^\tau(x_1, ..., x_{n+m+1}) = Q(x_1, ..., x_n, x_{n+\tau(1)}, ..., x_{n+\tau(m)}, x_{n+m+1})$, so $(Q^\tau)_{x_{n+\tau(i)}x_{n+m+1}} = (Q_{x_{n+i}x_{n+m+1}})^\tau$. Then

$$p_{n,m}(Q^\tau) = \frac{1}{m}\sum_{i=1}^m \mathrm{ad}\big((Q^\tau)_{x_{n+i}x_{n+m+1}}\big)(x_{n+i}) = \frac{1}{m}\sum_{i=1}^m \mathrm{ad}\big((Q^\tau)_{x_{n+\tau(i)}x_{n+m+1}}\big)(x_{n+\tau(i)})$$
$$= \frac{1}{m}\sum_{i=1}^m \mathrm{ad}\big((Q_{x_{n+i}x_{n+m+1}})^\tau\big)(x_{n+\tau(i)}) = (p_{n,m}(Q))^\tau,$$

which proves the $\mathfrak{S}_m$-equivariance.

Let us prove that the announced diagram commutes. Let $\lambda_0 \in \mathrm{LA}(T_{n'}, T_n)$ and $P \in \mathrm{LA}(T_n \otimes T_m \otimes \mathbf{id}, \mathbf{id})$. We must show that the images of $\lambda_0 \otimes P$ in $\mathrm{LA}(T_{n'} \otimes T_m, \mathbf{id})$ by two maps coincide. By linearity, we may assume that λ_0 has the form $x_1 \otimes .... \otimes x_{n'} \mapsto P_1(x_i, i \in f^{-1}(1)) \otimes ... \otimes P_n(x_i, i \in f^{-1}(n))$, where $f : [n'] \to [n]$ is a map and $P_i \in \mathrm{F}(x_{i'}, i' \in f^{-1}(i))$. The commutativity of the diagram then follows from the equality

$$P_{x_{n+i}x_{n+m+1}}\big(P_1(x_i, i \in f^{-1}(1)), ..., P_n(x_i, i \in f^{-1}(n)), x_{n'+1}, ..., x_{n'+m+1}\big)$$
$$= \Big(P(P_1(x_i, i \in f^{-1}(1)), ..., P_n(x_i, i \in f^{-1}(n)), x_{n'+1}, ..., x_{n'+m+1})\Big)_{x_{n'+i}x_{n'+m+1}}.$$

□

If $Z \in \mathrm{Irr}(\mathrm{Sch})$, we now define

$$\mu_Z \in \mathrm{LA}(Z \otimes \mathbf{id}, Z)$$

as follows. Let n be an integer ≥ 0. The decomposition $T_n = \oplus_{Z\in \mathrm{Irr}(\mathrm{Sch}),|Z|=n} Z \otimes \pi_Z$ gives rise to an isomorphism $\mathrm{LA}(T_n \otimes \mathbf{id}, T_n) \simeq \oplus_{Z,W\in \mathrm{Irr}(\mathrm{Sch}),|Z|=|W|=n} \mathrm{LA}(Z \otimes \mathbf{id}, W) \otimes \mathrm{Vect}(\pi_Z, \pi_W)$. On the other hand, μ_n has the $\mathfrak{S}_n$-equivariance property $\mu_n \circ (\sigma \otimes \mathrm{id}_{\mathbf{id}}) = \sigma \circ \mu_n$ for any $\sigma \in \mathfrak{S}_n$. It follows that μ_n decomposes as $\oplus_{Z\in \mathrm{Irr}(\mathrm{Sch}),|Z|=n} \mu_Z \otimes \mathrm{id}_{\pi_Z}$. This defines μ_Z for any $Z \in \mathrm{Irr}(\mathrm{Sch})$ with $|Z| = n$.

For $W, Z \in \mathrm{Irr}(\mathrm{Sch})$, define

$$i_{W,Z} : \mathrm{LA}(W \otimes Z, \mathbf{id}) \to \mathrm{LA}(W \otimes Z \otimes \mathbf{id}, \mathbf{id}), \quad \lambda \mapsto \lambda \circ (\mathrm{id}_W \otimes \mu_Z).$$

For $Z, W, W' \in \mathrm{Irr}(\mathrm{Sch})$, define

$$c_{W',W,Z} : \mathrm{LA}(W', W) \otimes \mathrm{LA}(W \otimes Z, \mathbf{id}) \to \mathrm{LA}(W' \otimes Z, \mathbf{id}), \quad \lambda_0 \otimes \lambda \mapsto \lambda \circ (\lambda_0 \otimes \mathrm{id}_Z).$$

Then the diagram

$$\begin{array}{ccc} \mathrm{LA}(W', W) \otimes \mathrm{LA}(W \otimes Z, \mathbf{id}) & \overset{\mathrm{id}\otimes i_{W,Z}}{\to} & \mathrm{LA}(W', W) \otimes \mathrm{LA}(W \otimes Z \otimes \mathbf{id}, \mathbf{id}) \\ {\scriptstyle c_{W',W,Z}}\downarrow & & \downarrow{\scriptstyle c_{W',W,Z\otimes\mathbf{id}}} \\ \mathrm{LA}(W' \otimes Z, \mathbf{id}) & \overset{i_{W',Z}}{\to} & \mathrm{LA}(W' \otimes Z \otimes \mathbf{id}, \mathbf{id}) \end{array}$$

commutes.

For $W, Z \in \mathrm{Irr}(\mathrm{Sch})$, define a linear map

$$p_{W,Z} : \mathrm{LA}(W \otimes Z \otimes \mathbf{id}, \mathbf{id}) \to \mathrm{LA}(W \otimes Z, \mathbf{id})$$

as follows. For n,m integers ≥ 0, the decompositions $T_n = \oplus_{W\in\mathrm{Irr}(\mathrm{Sch}),|W|=n} W\otimes\pi_W$, $T_m = \oplus_{Z\in\mathrm{Irr}(\mathrm{Sch}),|Z|=m} Z\otimes\pi_Z$ give rise to a decomposition

$$\begin{aligned}&\mathrm{Vect}(\mathrm{LA}(T_n\otimes T_m,\mathbf{id}),\mathrm{LA}(T_n\otimes T_m\otimes\mathbf{id},\mathbf{id}))\\ &\simeq \oplus_{\substack{W,W',Z,Z'||W|=|W'|=n,\\ |Z|=|Z'|=m}} \mathrm{Vect}(\mathrm{LA}(W\otimes Z,\mathbf{id}),\mathrm{LA}(W'\otimes Z'\otimes\mathbf{id},\mathbf{id}))\otimes\mathrm{Vect}(\pi_W\otimes\pi_Z,\pi_{W'}\otimes\pi_{Z'})\end{aligned}$$

which is $\mathfrak{S}_n\times\mathfrak{S}_m$-equivariant. Then $p_{n,m}\in\mathrm{Vect}(\mathrm{LA}(T_n\otimes T_m,\mathbf{id}),\mathrm{LA}(T_n\otimes T_m\otimes\mathbf{id},\mathbf{id}))$ is $\mathfrak{S}_n\times\mathfrak{S}_m$-invariant, which implies that it decomposes as $\sum_{W,Z\in\mathrm{Irr}(\mathrm{Sch})\,||W|=n,|Z|=m} p_{W,Z}\otimes\mathrm{id}_{\pi_W\otimes\pi_Z}$. This defines $p_{W,Z}$ for $W,Z\in\mathrm{Irr}(\mathrm{Sch})$.

Proposition 3.2. *We have $p_{W,Z}\circ i_{W,Z}=\mathrm{id}$, and the diagram*

$$\begin{array}{ccc}\mathrm{LA}(W',W)\otimes\mathrm{LA}(W\otimes Z\otimes\mathbf{id},\mathbf{id}) & \overset{\mathrm{id}\otimes p_{W,Z}}{\to} & \mathrm{LA}(W',W)\otimes\mathrm{LA}(W\otimes Z,\mathbf{id})\\ {\scriptstyle c_{W',W,Z\otimes\mathbf{id}}}\downarrow & & \downarrow{\scriptstyle c_{W',W,Z}}\\ \mathrm{LA}(W'\otimes Z\otimes\mathbf{id},\mathbf{id}) & \overset{p_{W',Z}}{\to} & \mathrm{LA}(W'\otimes Z,\mathbf{id})\end{array}\tag{22}$$

commutes.

Proof. This is obtained by taking the isotypic components of the statements of Lemma 3.1, and using that $i_{n,m}=\oplus_{W,Z\in\mathrm{Irr}(\mathrm{Sch})\,||W|=n,|Z|=m} i_{W,Z}\otimes\mathrm{id}_{\pi_W\otimes\pi_Z}$. □

We now prove:

Proposition 3.3. *The map $\mathrm{LBA}_f(\mathbf{id},\mathbf{id})\to\mathrm{LBA}_f(\wedge^2,\mathbf{id})$, $x\mapsto x\circ\mu$, is injective.*

Proof. Let $\alpha: C\to D$ be a morphism in LBA. We will prove that $i_\alpha:\mathrm{LBA}_\alpha(\mathbf{id},\mathbf{id})\to\mathrm{LBA}_\alpha(\wedge^2,\mathbf{id})\subset\mathrm{LBA}_\alpha(\mathbf{id}^{\otimes 2},\mathbf{id})$, $x\mapsto x\circ\mu$ is injective. For this, we will construct a map $p_\alpha:\mathrm{LBA}_\alpha(\mathbf{id}^{\otimes 2},\mathbf{id})\to\mathrm{LBA}_\alpha(\mathbf{id},\mathbf{id})$, such that $p_\alpha\circ i_\alpha=\mathrm{id}$.

The first map is the vertical cokernel of the commutative diagram

$$\begin{array}{ccc}\mathrm{LBA}(D\otimes\mathbf{id},\mathbf{id}) & \overset{i_D}{\to} & \mathrm{LBA}(D\otimes\mathbf{id}^{\otimes 2},\mathbf{id})\\ {\scriptstyle -\circ(\alpha\otimes\mathrm{id}_{\mathbf{id}})}\downarrow & & \downarrow{\scriptstyle -\circ(\alpha\otimes\mathrm{id}_{\mathbf{id}})}\\ \mathrm{LBA}(C\otimes\mathbf{id},\mathbf{id}) & \overset{i_C}{\to} & \mathrm{LBA}(C\otimes\mathbf{id}^{\otimes 2},\mathbf{id})\end{array}$$

where $i_X(x)=x\circ(\mathrm{id}_X\otimes\mu)$ for $X=C,D$.

We will construct a commutative diagram

$$\begin{array}{ccc}\mathrm{LBA}(D\otimes\mathbf{id}^{\otimes 2},\mathbf{id}) & \overset{p_D}{\to} & \mathrm{LBA}(D\otimes\mathbf{id},\mathbf{id})\\ {\scriptstyle -\circ(\alpha\otimes\mathrm{id}_{\mathbf{id}^{\otimes 2}})}\downarrow & & \downarrow{\scriptstyle -\circ(\alpha\otimes\mathrm{id}_{\mathbf{id}^{\otimes 2}})}\\ \mathrm{LBA}(C\otimes\mathbf{id}^{\otimes 2},\mathbf{id}) & \overset{p_C}{\to} & \mathrm{LBA}(C\otimes\mathbf{id},\mathbf{id})\end{array}\tag{23}$$

such that $p_C\circ i_C=\mathrm{id}$ and $p_D\circ i_D=\mathrm{id}$; then we will define p_α as the vertical cokernel of this diagram.

Set $A_C:=\mathrm{LBA}(C\otimes\mathbf{id},\mathbf{id})$, $A'_C:=\mathrm{LBA}(C\otimes\mathbf{id}^{\otimes 2},\mathbf{id})$. Let us study the map

$$i_C: A_C\to A'_C.$$

We have $A_C=\oplus_{W,Z\in\mathrm{Irr}(\mathrm{Sch})}A_C(W,Z)$ and $A'_C=\oplus_{W,Z',Z''\in\mathrm{Irr}(\mathrm{Sch})}A'_C(W,Z',Z'')$, where

$$A_C(W,Z):=\mathrm{LCA}(C,W)\otimes\mathrm{LCA}(\mathbf{id},Z)\otimes\mathrm{LA}(W\otimes Z,\mathbf{id}),$$

$$A'_C(W,Z',Z''):=\mathrm{LCA}(C,W)\otimes\mathrm{LCA}(\mathbf{id},Z')\otimes\mathrm{LCA}(\mathbf{id},Z'')\otimes\mathrm{LA}(W\otimes Z'\otimes Z'',\mathbf{id}).$$

Set $A''_C:=\oplus_{W,Z\in\mathrm{Irr}(\mathrm{Sch})}A'_C(W,Z,\mathbf{id})$. We have a natural projection map $A'_C\to A''_C$.

Then the composition $A_C \stackrel{i_C}{\to} A'_C \to A''_C$ is the direct sum over W, Z of the maps $A_C(W,Z) \to A'_C(W,Z,\mathbf{id})$, given by

$$\mathrm{LCA}(C,W) \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LA}(W \otimes Z, \mathbf{id}) \to \mathrm{LCA}(C,W) \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LCA}(\mathbf{id},\mathbf{id}) \otimes \mathrm{LA}(W \otimes Z \otimes \mathbf{id}, \mathbf{id}), \quad \kappa_C \otimes \kappa_{\mathbf{id}} \otimes \lambda \mapsto \kappa_C \otimes \kappa_{\mathbf{id}} \otimes 1 \otimes i_{W,Z}(\lambda).$$

Define the map

$$p_C : A'_C \to A_C$$

as the composition $A'_C \to A''_C \to A_C$, where the first map is the natural projection and the second map is the direct sum over W, Z of the maps $A_C(W,Z) \to A'_C(W,Z,\mathbf{id})$, given by

$$\mathrm{LCA}(C,W) \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LCA}(\mathbf{id},\mathbf{id}) \otimes \mathrm{LA}(W \otimes Z \otimes \mathbf{id}, \mathbf{id}) \to \mathrm{LCA}(C,W) \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LA}(W \otimes Z, \mathbf{id}), \quad \kappa_C \otimes \kappa_{\mathbf{id}} \otimes 1 \otimes \lambda' \mapsto \kappa_C \otimes \kappa_{\mathbf{id}} \otimes p_{W,Z}(\lambda').$$

Then $p_{W,Z} \circ i_{W,Z} = \mathrm{id}$ implies that $p_C \circ i_C = \mathrm{id}$.

Let us prove that (23) commutes. For this, we will prove that

$$\begin{array}{ccc} \mathrm{LBA}(C,D) \otimes \mathrm{LBA}(D \otimes \mathbf{id}^{\otimes 2}, \mathbf{id}) & \stackrel{\mathrm{id} \otimes p_D}{\to} & \mathrm{LBA}(C,D) \otimes \mathrm{LBA}(D \otimes \mathbf{id}, \mathbf{id}) \\ \downarrow & & \downarrow \\ \mathrm{LBA}(C \otimes \mathbf{id}^{\otimes 2}, \mathbf{id}) & \stackrel{p_D}{\to} & \mathrm{LBA}(C \otimes \mathbf{id}, \mathbf{id}) \end{array}$$

commutes, where the vertical maps are $\alpha \otimes x \mapsto x \circ (\alpha \otimes \mathrm{id}_{\mathbf{id}^{\otimes 2}})$ (right map) and $\alpha \otimes x \mapsto x \circ (\alpha \otimes \mathrm{id}_{\mathbf{id}})$ (left map).

This diagram is the same as

$$\begin{array}{ccc} \begin{array}{c} \oplus_{U,W,Z',Z'' \in \mathrm{Irr(Sch)}} \mathrm{LCA}(C,U) \otimes \mathrm{LA}(U,D) \\ \otimes \mathrm{LCA}(D,W) \otimes \mathrm{LCA}(\mathbf{id},Z') \otimes \mathrm{LCA}(\mathbf{id},Z'') \\ \otimes \mathrm{LA}(W \otimes Z' \otimes Z'', \mathbf{id}) \end{array} & \stackrel{(1)}{\to} & \begin{array}{c} \oplus_{U,W,Z \in \mathrm{Irr(Sch)}} \mathrm{LCA}(C,U) \\ \otimes \mathrm{LA}(U,D) \otimes \mathrm{LCA}(D,W) \\ \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LA}(W \otimes Z, \mathbf{id}) \end{array} \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (4)} \\ \begin{array}{c} \oplus_{V,Z',Z'' \in \mathrm{Irr(Sch)}} \mathrm{LCA}(C,V) \otimes \mathrm{LCA}(\mathbf{id},Z') \\ \otimes \mathrm{LCA}(\mathbf{id},Z'') \otimes \mathrm{LA}(V \otimes Z' \otimes Z'', \mathbf{id}) \end{array} & \stackrel{(2)}{\to} & \begin{array}{c} \oplus_{V,Z \in \mathrm{Irr(Sch)}} \mathrm{LCA}(C,V) \\ \otimes \mathrm{LCA}(\mathbf{id},Z) \otimes \mathrm{LA}(V \otimes Z, \mathbf{id}) \end{array} \end{array}$$

where (1) is zero on the components with $Z'' \neq \mathbf{id}$; it takes the component $(U,W,Z,\mathbf{id})$ to the component (U,W,Z) by the map $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes 1 \otimes p_{W,Z}$;

(2) is zero on the components with $Z'' \neq \mathbf{1}$; its takes the component $(U,Z,\mathbf{id})$ to the component (U,Z) by the map $\mathrm{id} \otimes 1 \otimes p_{U,Z}$;

(3) is the composition of the natural map

$$\mathrm{LA}(U,D) \otimes \mathrm{LCA}(D,W) \to \mathrm{LBA}(U,W) \simeq \oplus_{V \in \mathrm{Irr(Sch)}} \mathrm{LCA}(U,V) \otimes \mathrm{LA}(V,W),$$

of the composition $\mathrm{LCA}(C,U) \otimes \mathrm{LCA}(U,V) \to \mathrm{LCA}(C,V)$ and of the map $\mathrm{LA}(V,W) \otimes \mathrm{LA}(W \otimes Z' \otimes Z'', \mathbf{id}) \to \mathrm{LA}(V, \otimes Z' \otimes Z'', \mathbf{id})$, $\alpha \otimes \beta \mapsto \beta \circ (\alpha \otimes \mathrm{id}_{Z' \otimes Z''})$;

(4) is the composition of same maps, where in the last step $Z' \otimes Z''$ is replaced by Z.

The commutativity of the diagram formed by these maps then follows from that of (22). $\square$

4. Quantization functors

4.1. **Definition.** A quantization functor is a prop morphism $Q : \mathrm{Bialg} \to S(\mathbf{LBA})$, such that:

(a) the composed morphism $\mathrm{Bialg} \stackrel{Q}{\to} S(\mathbf{LBA}) \to S(\mathbf{Sch})$ (where the second morphism is given by the specialization $\mu = \delta = 0$) is the propic version of the bialgebra structure of the symmetric algebras $S(V)$, where the elements of V are primitive, and

(b) (classical limits) $\mathrm{pr}_1 \circ Q(m) \circ (\mathrm{inj}_1^{\otimes 2} \circ \mathrm{can}) \in \mathbf{LBA}(\wedge^2, \mathbf{id})$ has the form μ + terms of positive δ-degree, and $(\mathrm{Alt} \circ \mathrm{pr}_1^{\otimes 2}) \circ Q(\Delta) \circ \mathrm{inj}_1 \in \mathbf{LBA}(\mathbf{id}, \wedge^2) = \delta$ + terms of positive μ-degree.

Here $\mathrm{inj}_1 : \mathbf{id} \to S$ and $\mathrm{pr}_1 : S \to \mathbf{id}$ are the canonical injection and projection maps, and $\mathrm{inc} : \wedge^2 \to T_2$, $\mathrm{Alt} : T_2 \to \wedge^2$ are the inclusion and alternation maps.

Note that (a) implies that $Q(\eta) = \mathrm{inj}_0 \in \mathbf{LBA}(\mathbf{1}, S)$, and $Q(\varepsilon) = \mathrm{pr}_0 \in \mathbf{LBA}(S, \mathbf{1})$, where $\mathrm{inj}_0 : \mathbf{1} \to S$ and $\mathrm{pr}_0 : S \to \mathbf{1}$ are the natural injection and projection.

The quantization functors Q, Q' are called equivalent iff there exists an inner automorphism $\theta(\xi_0)$ of $S(\mathbf{LBA})$, such that $Q' = \theta(\xi_0) \circ Q$.

4.2. **Construction of quantization functors.** In [EK1], Etingof and Kazhdan constructed a quantization functor corresponding to each associator Φ. This construction can be described as follows ([Enr3]).

Let $\mathfrak{t}_n$ be the Lie algebra with generators t_{ij}, $1 \leq i \neq j \leq n$ and relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$, $[t_{ij}, t_{kl}] = 0$ (i, j, k, l distinct). It is graded by $\deg(t_{ij}) = 1$. We have a graded algebra morphism $U(\mathfrak{t}_n) \to \mathbf{U}_n$, taking t_{ij} to $t^{i,j}$, where $t \in \mathbf{U}_2$ is $r + r^{2,1}$.

The family $(\mathfrak{t}_n)_{n \geq 0}$ is a $\mathcal{C}$-Lie algebra, and $U(\mathfrak{t}_n) \to \mathbf{U}_n$ is a morphism of $\mathcal{C}$-algebras.

An associator Φ is an element of $\widehat{U(\mathfrak{t}_3)}_1^\times$, satisfying certain relations (see [Dr3], where it is proved that associators exist over $\mathbf{k}$). We fix an associator Φ; we also denote by Φ its image in $(\widehat{\mathbf{U}}_3)_1^\times$.

One constructs $\mathrm{J} \in (\widehat{\mathbf{U}}_2)_1^\times$, such that $\mathrm{J} = 1 - r/2 + ...$, and

$$\mathrm{J}^{1,2}\, \mathrm{J}^{12,3} = \mathrm{J}^{2,3}\, \mathrm{J}^{1,23}\, \Phi. \tag{24}$$

Then one sets

$$\mathrm{R} := \mathrm{J}^{2,1}\, e^{t/2}\, \mathrm{J}^{-1} \in (\widehat{\mathbf{U}}_2)_1^\times.$$

Using J and R, we will define elements of the quasi-bi-multiprop $\mathbf{\Pi}$.

We define

$$\Delta_\Pi \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\boxtimes 2}), \quad \mathrm{Ad}(\mathrm{J}) \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 2}, (S \underline{\boxtimes} S)^{\boxtimes 2}).$$

One checks that the elements $m_\Pi^{(2)} \boxtimes m_\Pi^{(2)} \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 6}, (S \underline{\boxtimes} S)^{\boxtimes 2})$, $(142536) \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 6}, (S \underline{\boxtimes} S)^{\boxtimes 6})$ and $\mathrm{J} \boxtimes \mathrm{id}_{(S \underline{\boxtimes} S)^{\boxtimes 2}} \boxtimes \mathrm{J}^{-1} \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 2}, (S \underline{\boxtimes} S)^{\boxtimes 6})$ are composable, and we set

$$\mathrm{Ad}(\mathrm{J}) := (m_\Pi^{(2)} \boxtimes m_\Pi^{(2)}) \circ (142536) \circ \big(\mathrm{J} \boxtimes \mathrm{id}_{(S \underline{\boxtimes} S)^{\boxtimes 2}} \boxtimes \mathrm{J}^{-1}\big) \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 2}, (S \underline{\boxtimes} S)^{\boxtimes 2}).$$

A graph for this element is as follows. Set $F_1 = ... = G'_2 = S$, then this is an element of $\mathbf{\Pi}((F_1 \boxtimes F_2) \underline{\boxtimes} (G_1 \boxtimes G_2), (F'_1 \boxtimes F'_2) \underline{\boxtimes} (G'_1 \boxtimes G'_2))$, and the edges are $F_i \to F'_j$, $G'_i \to G_j$, $G'_j \to F'_i$ $(i, j = 1, 2)$.

Now $\mathrm{Ad}(\mathrm{J})$ and Δ_0 can be composed, and we set

$$\Delta_\Pi := \mathrm{Ad}(\mathrm{J}) \circ \Delta_0 \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\boxtimes 2}).$$

A graph for this element is as follows. If we set $F = ... = G'_2 = S$, then this is an element of $\mathbf{\Pi}(F \underline{\boxtimes} G, (F'_1 \boxtimes F'_2) \underline{\boxtimes} (G'_1 \boxtimes G'_2))$. The vertices are then $F \to F'_i$, $G'_i \to G$, $G'_i \to F'_j$ $(i, j = 1, 2)$.

The elements m_Π, Δ_Π then satisfy (15); moreover, the following elements make sense, and the identities hold:

$$\Delta_\Pi \circ m_\Pi = (m_\Pi \boxtimes m_\Pi) \circ (1324) \circ (\Delta_\Pi \boxtimes \Delta_\Pi), \quad (\Delta_\Pi \boxtimes \mathrm{id}_{S \underline{\boxtimes} S}) \circ \Delta_\Pi = (\mathrm{id}_{S \underline{\boxtimes} S} \boxtimes \Delta_\Pi) \circ \Delta_\Pi. \tag{25}$$

In particular, $\overline{m}_\Pi := \Delta_\Pi^* \circ (21) \in \mathbf{\Pi}((S \underline{\boxtimes} S)^{\boxtimes 2}, S \underline{\boxtimes} S)$ and $\overline{\Delta}_\Pi := m_\Pi^* \in \mathbf{\Pi}(S \underline{\boxtimes} S, (S \underline{\boxtimes} S)^{\boxtimes 2})$ satisfy relations (15), (25).

Moreover, $\mathrm{R} \in \mathbf{\Pi}(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (S\underline{\boxtimes}S)^{\boxtimes 2})$ satisfies the quasitriangular identities

$$\left(\Delta_\Pi \boxtimes \mathrm{id}_{S\underline{\boxtimes}S}\right) \circ \mathrm{R} = \left(\mathrm{id}_{(S\underline{\boxtimes}S)^{\boxtimes 2}} \boxtimes m_\Pi\right) \circ (1324) \circ (\mathrm{R} \boxtimes \mathrm{R}) \tag{26}$$

and

$$\left(\mathrm{id}_{S\underline{\boxtimes}S} \boxtimes \Delta_\Pi\right) \circ \mathrm{R} = (132) \circ \left(m_\Pi \boxtimes \mathrm{id}_{(S\underline{\boxtimes}S)^{\boxtimes 2}}\right) \circ (1324) \circ (\mathrm{R} \boxtimes \mathrm{R}). \tag{27}$$

Define

$$\ell := \left(\mathrm{id}_{S\underline{\boxtimes}S} \boxtimes \mathrm{can}^*_{S\underline{\boxtimes}S}\right) \circ \left(\mathrm{R} \boxtimes \mathrm{id}_{S\underline{\boxtimes}S}\right) \in \mathbf{\Pi}(S\underline{\boxtimes}S, S\underline{\boxtimes}S).$$

The following proposition is a consequence of the quasitriangular identities (26), (27):

Proposition 4.1. *The following elements are defined, and the equations hold:*

$$m_\Pi \circ \ell^{\boxtimes 2} = \ell \circ \overline{m}_\Pi, \quad \Delta_\Pi \circ \ell = \ell^{\boxtimes 2} \circ \overline{\Delta}_\Pi.$$

The flatness statement of [Enr3] can be restated as follows:

Proposition 4.2. *There exist elements* $\mathrm{R}_+ \in \mathbf{\Pi}(S\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}S)$ *and* $\mathrm{R}_- \in \mathbf{\Pi}(\mathbf{1}\underline{\boxtimes}S, S\underline{\boxtimes}S)$*, such that*

$$\mathrm{R} = (\mathrm{R}_+ \boxtimes \mathrm{R}_-) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}. \tag{28}$$

Moreover, $\mathrm{R}_\pm$ *are right-invertible, i.e., there exist* $\mathrm{R}_+^{(-1)} \in \mathbf{\Pi}(S\underline{\boxtimes}S, S\underline{\boxtimes}\mathbf{1})$ *and* $\mathrm{R}_-^{(-1)} \in \mathbf{\Pi}(S\underline{\boxtimes}S, \mathbf{1}\underline{\boxtimes}S)$*, with graphs* $F \to F'$ *and* $G' \to G$ *[where* $\mathrm{R}_+^{(-1)}$ *is viewed as an element of* $\mathbf{\Pi}(F\underline{\boxtimes}G, F'\underline{\boxtimes}\mathbf{1})$ *and* $\mathrm{R}_-^{(-1)}$ *as an element of* $\mathbf{\Pi}(F\underline{\boxtimes}G, \mathbf{1}\underline{\boxtimes}G')$*], such that* $\mathrm{R}_+^{(-1)} \circ \mathrm{R}_+ = \mathrm{id}_{S\underline{\boxtimes}\mathbf{1}}$ *and* $\mathrm{R}_-^{(-1)} \circ \mathrm{R}_- = \mathrm{id}_{\mathbf{1}\underline{\boxtimes}S}$ *(where the compositions are well-defined).*

Notice that $(\mathrm{R}_+, \mathrm{R}_-)$ is uniquely defined only up to a transformation

$$(\mathrm{R}_+, \mathrm{R}_-) \to (\mathrm{R}_+ \circ \mathrm{R}'', \mathrm{R}_- \circ ((\mathrm{R}'')^*)^{-1}),$$

where $\mathrm{R}'' \in \mathbf{\Pi}(S\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}\mathbf{1})^\times$. This transformation will not change the equivalence class of Q.

(28) implies that

$$\ell = \mathrm{R}_+ \circ \mathrm{R}_-^*.$$

Proposition 4.3. *The following elements are defined, and the equations hold:*

$$\mathrm{R}_+^{(-1)} \circ m_\Pi \circ \mathrm{R}_+^{\boxtimes 2} = \mathrm{R}_-^* \circ \overline{m}_\Pi \circ (\mathrm{R}_-^{(-1)*})^{\boxtimes 2}, \quad (\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ \Delta_\Pi \circ \mathrm{R}_+ = (\mathrm{R}_-^*)^{\boxtimes 2} \circ \overline{\Delta}_\Pi \circ (\mathrm{R}_-^{(-1)})^*. \tag{29}$$

Let $m_a \in \mathbf{\Pi}(S\underline{\boxtimes}\mathbf{1}, (S \boxtimes S)\underline{\boxtimes}\mathbf{1})$ *be the value of both sides of the first identity of (29), and let* $\Delta_a \in \mathbf{\Pi}((S \boxtimes S)\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}\mathbf{1})$ *be the common value of both sides of the second identity. Then* m_a, Δ_a *satisfy (15) and (25).*

Then there is a unique morphism $Q : \mathrm{Bialg} \to S(\mathbf{LBA})$*, such that* $Q(m) =$ *the element of* $\mathbf{LBA}(S^{\otimes 2}, S)$ *corresponding to* m_a*,* $Q(\Delta) =$ *the element of* $\mathbf{LBA}(S, S^{\otimes 2})$ *corresponding to* Δ_a*,* $Q(\varepsilon) =$ *the element of* $\mathbf{LBA}(S, \mathbf{1})$ *corresponding to* $1 \in \mathbf{k}$*,* $Q(\eta) =$ *the element of* $\mathbf{LBA}(\mathbf{1}, S)$ *corresponding to* $1 \in \mathbf{k}$*.*

Proof. The proof follows that of the following statement: let $\mathcal{S}$ be a symmetric tensor category, let $A, X, B \in \mathrm{Ob}(\mathcal{S})$. Assume that $m_A \in \mathcal{S}(A^{\otimes 2}, A)$, $\Delta_A \in \mathcal{S}(A, A^{\otimes 2})$,... is a bialgebra structure on A in the category $\mathcal{C}$. Let similarly $(m_X, \Delta_X, ...)$ and $(m_B, \Delta_B, ...)$ be $\mathcal{S}$-bialgebra structures on X and B. Let $\ell_{AX} \in \mathcal{S}(A, X)$ and $\ell_{XB} \in \mathcal{S}(X, B)$ be morphisms of $\mathcal{S}$-bialgebras, such that ℓ_{AX} is right invertible and ℓ_{XB} is left invertible: let $\ell_{XA} \in \mathcal{S}(X, A)$ and $\ell_{BX} \in \mathcal{S}(X, B)$ be such that $\ell_{AX} \circ \ell_{XA} = \mathrm{id}_X$, and $\ell_{BX} \circ \ell_{XB} = \mathrm{id}_X$. Then $\ell_{AX} \circ m_A \circ \ell_{XA}^{\otimes 2} = \ell_{BX} \circ m_B \circ \ell_{XB}^{\otimes 2}$, $\ell_{AX}^{\otimes 2} \circ \Delta_A \circ \ell_{XA} = \ell_{BX}^{\otimes 2} \circ \Delta_B \circ \ell_{XB}$, etc. If we call $m_X \in \mathcal{S}(X^{\otimes 2}, X)$ (resp., $\Delta_X \in \mathcal{S}(X, X^{\otimes 2})$, etc.) the common value of both sides of the first (resp., second) identity, then $(m_X, \Delta_X, ...)$ is a $\mathcal{S}$-bialgebra structure on X . $\square$

According to [Enr3], J is uniquely determined by (24) only up to a gauge transformation $\mathrm{J} \mapsto {}^u\,\mathrm{J} = u^1 u^2\, \mathrm{J} (u^{12})^{-1}$, where $u \in (\widehat{\mathbf{U}}_1)_1^\times$.

Lemma 4.4. *Quantization functors corresponding to* J *and to* ${}^u\,\mathrm{J}$ *are equivalent.*

Proof. We have $u, u^{-1} \in \widehat{\mathbf{U}}_1 \simeq \mathbf{\Pi}(\mathbf{1} \underline{\boxtimes} \mathbf{1}, S \underline{\boxtimes} S)$. Let us set

$$\mathrm{Ad}(u) := m_\Pi^{(2)}(u \boxtimes \mathrm{id}_{S \underline{\boxtimes} S} \boxtimes u^{-1}) \in \mathbf{\Pi}(S \underline{\boxtimes} S, S \underline{\boxtimes} S)^\times$$

(one checks that the r.h.s. makes sense).

Let us view $\mathrm{Ad}(u)$ as an element of $\mathbf{\Pi}(F \underline{\boxtimes} G, F' \underline{\boxtimes} G')$, then a graph for $\mathrm{Ad}(u)$ and $\mathrm{Ad}(u)^{-1} (= \mathrm{Ad}(u^{-1}))$ is $F \to F'$, $G' \to G$, $G' \to F'$.

In the same way, $\mathrm{Ad}(u)^*, (\mathrm{Ad}(u)^{-1})^* \in \mathbf{\Pi}(S \underline{\boxtimes} S, S \underline{\boxtimes} S)^\times$, and a graph for these elements is $F \to F'$, $G' \to G$, $F \to G$.

Let us denote by ${}^u\,\mathrm{R}, {}^u\ell, ..., {}^uQ$ the analogues of $\mathrm{R}, \ell, ..., Q$, with J replaced by ${}^u\,\mathrm{J}$. These analogues can be expressed as follows: ${}^u m_\Pi = m_\Pi$, ${}^u\Delta_\Pi = \mathrm{Ad}(u)^{\boxtimes 2} \circ \Delta_\Pi \circ \mathrm{Ad}(u)^{-1}$ (one checks that the r.h.s. is well-defined), ${}^u\,\mathrm{R} = u^1 u^2\, \mathrm{R}(u^1 u^2)^{-1}$, therefore ${}^u\ell = \mathrm{Ad}(u) \circ \ell \circ \mathrm{Ad}(u)^*$ (one checks that the r.h.s. is well-defined). We then make the following choices for ${}^u\,\mathrm{R}_\pm$: ${}^u\,\mathrm{R}_\pm = \mathrm{Ad}(u) \circ \mathrm{R}_\pm$ (one checks that both r.h.s. are well-defined).

We then have ${}^u\,\mathrm{R}_\pm^{(-1)} = \mathrm{R}_\pm^{(-1)} \circ \mathrm{Ad}(u)^{-1}$. Then:

$${}^u m_a = ({}^u\,\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ {}^u m_\Pi \circ {}^u\,\mathrm{R}_+ = (\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ (\mathrm{Ad}(u)^{-1})^{\boxtimes 2} \circ m_\Pi \circ \mathrm{Ad}(u) \circ \mathrm{R}_+ = (\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ m_\Pi \circ \mathrm{R}_+ = m_a,$$

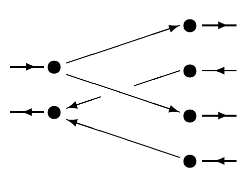
and

$${}^u\Delta_a = ({}^u\,\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ {}^u\Delta_\Pi \circ {}^u\,\mathrm{R}_+ = (\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ (\mathrm{Ad}(u)^{-1})^{\boxtimes 2} \circ {}^u\Delta_\Pi \circ \mathrm{Ad}(u) \circ \mathrm{R}_+ = (\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ \Delta_\Pi \circ \mathrm{R}_+ = \Delta_a,$$

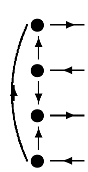
so ${}^uQ(m) = Q(m)$ and ${}^uQ(\Delta) = Q(\Delta)$, so ${}^uQ = Q$. □

Here are pictures of the main graphs of the above construction. The object S is represented by black vertices, and the object $\mathbf{1} \in \mathbf{Sch}_{(1)}$ is represented by white vertices.

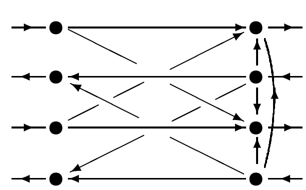
$\mathbf{\Delta_0}$



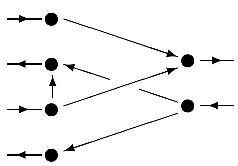
J



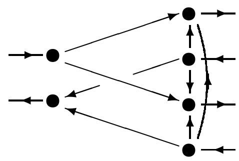
Ad(J)



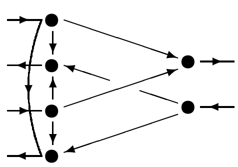
$\boldsymbol{m_\Pi}$



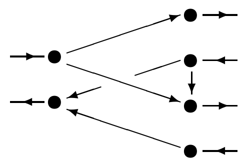
$\boldsymbol{\Delta_\Pi}$



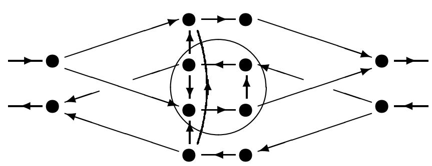
$\boldsymbol{\overline{m}_\Pi}$



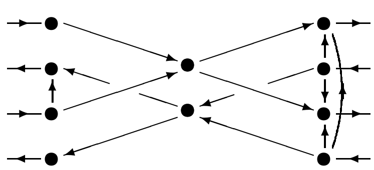
$\boldsymbol{\overline{\Delta}_\Pi}$



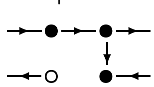
$\boldsymbol{m_\Pi \circ \Delta_\Pi}$ a priori undefined



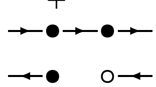
$\boldsymbol{\Delta_\Pi \circ m_\Pi}$ is defined



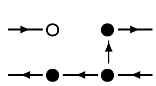
$\mathbf{R}_+$



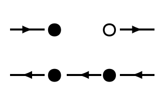
$\mathbf{R}_+^{(-1)}$



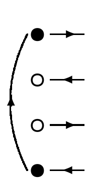
$\mathbf{R}_-$



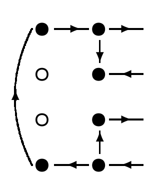
$\mathbf{R}_-^{(-1)}$

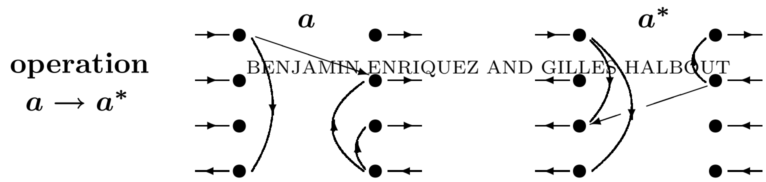


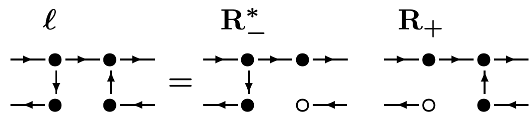
$\mathbf{can}_{S\boxtimes\mathbf{1}}$



$\mathbf{R}$







5. Compatibility of quantization functors with twists

5.1. **The category $\mathcal{Y}$.** We define $\mathcal{Y}$ as the category where objects are integer numbers ≥ 0, and $\mathcal{Y}(n,m)$ is the set of pairs (ϕ, o), where $\phi : [m] \to [n]$ is a partially defined function and $o = (o_1, ..., o_n)$, where o_i is a total order on $\phi^{-1}(i)$. If $(\phi, o) \in \mathcal{Y}(n,m)$ and $(\phi', o') \in \mathcal{Y}(m,p)$, then their composition is $(\phi'', o'') \in \mathcal{Y}(n,p)$, where $\phi'' = \phi \circ \phi'$ and $o'' = (o''_1, ..., o''_n)$, where o''_i is te lexicographic order on $(\phi'')^{-1}(i) = \sqcup_{j \in \phi^{-1}(i)} (\phi')^{-1}(j)$.

A $\mathcal{Y}$-vector space, (resp., a $\mathcal{Y}$-algebra) is a functor $\mathcal{Y} \to \mathrm{Vect}$ (resp., $\mathcal{Y} \to \mathrm{Alg}$). The forgetful morphism $\mathcal{Y} \to \mathcal{X}$ gives rise to functors $\{\mathcal{X}\text{-vector spaces}\} \to \{\mathcal{Y}\text{-vector spaces}\}$ and $\{\mathcal{X}\text{-algebras}\} \to \{\mathcal{Y}\text{-algebras}\}$. A $\mathcal{Y}$-vector space is therefore a collection of vector spaces $(V_n)_{n \geq 0}$ and of maps $V_n \to V_m$, $x \mapsto x^{\phi,o}$ for $(\phi, o) \in \mathcal{Y}(n,m)$.

If H is a (non-necessarily cocommutative) coalgebra (resp., bialgebra), then $(H^{\otimes n})_{n \geq 0}$ is a $\mathcal{Y}$-vector space (resp., $\mathcal{Y}$-algebra).

5.2. **$\mathcal{Y}$-algebra structures on $\widehat{\mathbf{U}}_n$, $\widehat{\mathbf{U}}_{n,f}$ associated with J.** A solution J of (24) gives rise to a $\mathcal{Y}$-algebra structures on $(\widehat{\mathbf{U}}_n)_{n \geq 0}$, $(\widehat{\mathbf{U}}_{n,f})_{n \geq 0}$, which we now define (we will call them the J-twisted structures).

For $(\phi, o) \in \mathcal{Y}(n,m)$, define $\mathrm{J}^{\phi,o} \in \widehat{\mathbf{U}}_n^\times$ as follows. For $\psi : [k] \to [m]$ an injective map, we set

$$\mathrm{J}_\psi = \mathrm{J}^{\psi(1),\psi(2)} ... \mathrm{J}^{\psi(1)...\psi(k-2),\psi(k-1)} \, \mathrm{J}^{\psi(1)...\psi(k-1),\psi(k)}$$

and

$$\mathrm{J}^{\phi,o} = \mathrm{J}_{\psi_1} ... \mathrm{J}_{\psi_n},$$

where $\psi_i : [|\phi^{-1}(i)|] \to \phi^{-1}$ is the unique order-preserving bijection.

The $\mathcal{Y}$-vector space structure on $(\mathbf{U}_n)_{n \geq 0}$ is then defined by $x \mapsto (x)_{\mathrm{J}}^{\phi,o} := (x)_{\mathrm{J}}^{\phi,o} := \mathrm{J}^{\phi,o} x^\phi (\mathrm{J}^{\phi,o})^{-1}$; the algebra structure is unchanged.

In the case of $\widehat{\mathbf{U}}_{n,f}$, the $\mathcal{Y}$-algebra structure is defined by $x \mapsto (x)_{\mathrm{J}}^{\phi,o} := \kappa_1^\Pi(\mathrm{J}^{\phi,o}) x^\phi \kappa_1^\Pi(\mathrm{J}^{\phi,o})^{-1}$.

Both $(\widehat{\mathbf{U}}_n)_{n \geq 0}$ and $(\widehat{\mathbf{U}}_{n,f})_{n \geq 0}$ are $\mathcal{Y}$-algebras, equipped with decreasing $\mathcal{Y}$-algebra filtrations (where the Nth step consists of the elements of degree $\geq N$).

5.3. **$\mathcal{Y}$-algebra structure on $\mathbf{P}(\mathbf{1}, S^{\otimes n})$.** Let $\mathbf{P}$ be a topological prop and let $\overline{Q} : \mathrm{Bialg} \to S(\mathbf{P})$ be a prop morphism. Recall that if H is a coalgebra, then $(H^{\otimes n})_{n \geq 0}$ is a $\mathcal{Y}$-vector space. Let us denote by $\Delta_H^{\phi,o} : H^{\otimes n} \to H^{\otimes m}$ the map corresponding to $(\phi, o) \in \mathcal{Y}(n,m)$. The propic versions of the maps $\Delta_H^{\phi,o}$ are elements $\Delta^{\phi,o} \in \mathrm{Coalg}(T_n, T_m)$, where Coalg is the prop of algebras (with generators Δ, η with the same relations as in Bialg). We also denote by $\Delta^{\phi,o} \in \mathrm{Bialg}(T_n, T_m)$ the images of these elements under the prop morphism $\mathrm{Coalg} \to \mathrm{Bialg}$.

Then $(\mathbf{P}(\mathbf{1}, S^{\otimes n}))_{n \geq 0}$ is a $\mathcal{Y}$-vector space: the map $\mathbf{P}(\mathbf{1}, S^{\otimes n}) \to \mathbf{P}(\mathbf{1}, S^{\otimes m})$ corresponding to $(\phi, o) \in \mathcal{D}(n,m)$ is $x \mapsto (x)_{\overline{Q}}^{\phi,o} := \overline{Q}(\Delta^{\phi,o}) \circ x$.

Each $\mathbf{P}(\mathbf{1}, S^{\otimes n})$ is equipped with the algebra structure

$$x \otimes y \mapsto x *_{\overline{Q}} y := \overline{Q}(m)^{\otimes n} \circ (1, n+1, 2, n+2, ...) \circ (x \otimes y).$$

The unit for this algebra is $\overline{Q}(\eta^{\otimes n})$.

Then this family of algebra structures is compatible with the $\mathcal{Y}$-structure, so $(\mathbf{P}(\mathbf{1}, S^{\otimes n}))_{n\geq 0}$ is a $\mathcal{Y}$-algebra.

In particular, the morphism $\kappa_1^{\Pi} \circ Q : \text{Bialg} \to S(\mathbf{LBA}_f)$ gives rise to a $\mathcal{Y}$-algebra structure on $\mathbf{LBA}_f(\mathbf{1}, S^{\otimes n})$. Using the identification $\mathbf{LBA}_f(\mathbf{1}, S^{\otimes n}) \simeq \mathbf{\Pi}_f(\mathbb{1}\underline{\boxtimes}\mathbb{1}, (S\underline{\boxtimes}\mathbb{1})^{\otimes n})$, the algebra structure is given by

$$x *_Q y := \kappa_1^{\Pi}(m_a)^{\boxtimes n} \circ (1, n+1, 2, n+2, ...) \circ (x \boxtimes y),$$

and the $\mathcal{Y}$-vector space structure by

$$(x)_Q^{\phi,o} := \kappa_1^{\Pi}(\Delta_a^{\phi,o}) \circ x.$$

5.4. **A $\mathcal{Y}$-algebra morphism $I_n : \mathbf{LBA}_f(\mathbf{1}, S^{\otimes n}) \to \widehat{\mathbf{U}}_{n,f}$.** Define a linear map

$$I_n : \mathbf{LBA}_f(\mathbf{1}, S^{\otimes n}) \simeq (\widehat{S^{\otimes n} \boxtimes \mathbf{1}})_f \simeq \mathbf{\Pi}_f(\mathbb{1}\underline{\boxtimes}\mathbb{1}, (S\underline{\boxtimes}\mathbb{1})^{\otimes n}) \to \widehat{\mathbf{U}}_{n,f},$$

$$\mathbf{\Pi}_f(\mathbb{1}\underline{\boxtimes}\mathbb{1}, (S\underline{\boxtimes}\mathbb{1})^{\otimes n}) \ni x \mapsto \kappa_1^{\Pi}(\mathrm{R}_+)^{\boxtimes n} \circ x.$$

This is a morphism of $\mathcal{Y}$-algebras, where $\mathbf{LBA}_f(\mathbf{1}, S^{\otimes n})$ is equipped with the structure corre-ponding to $S(\kappa_1) \circ Q$ and $\widehat{\mathbf{U}}_{n,f}$ is equipped with its J-twisted structure.

Then I_n is a filtered map, and the associated graded is the inclusion $\mathrm{LBA}_f(\mathbf{1}, S^{\otimes n}) \simeq (S^{\otimes n} \boxtimes \mathbf{1})_f \hookrightarrow (\Delta(S^{\otimes n}))_f \simeq \Pi_f(\mathbb{1}\underline{\boxtimes}\mathbb{1}, (S\underline{\boxtimes}S)^{\otimes n}) \simeq \mathbf{U}_{n,f}$.

5.5. **Construction of (v, F).**

Theorem 5.1. *There exists a pair (v, F), where $v \in (\widehat{\mathbf{U}}_{1,f})_1^{\times}$ and $\mathrm{F} \in ((\widehat{S^{\otimes 2} \boxtimes \mathbf{1}})_f)_1^{\times}$, such that*

$$\mathrm{J}(r+f) = v^1 v^2 I_2(\mathrm{F}) \mathrm{J}(r)(v^{12})^{-1} \tag{30}$$

(equality in $\widehat{\mathbf{U}}_{2,f}$, where v^{12} is defined using the $\mathcal{C}$-algebra structure on $\widehat{\mathbf{U}}_{n,f}$).

Then

$$(\mathrm{F})_Q^{1,2} *_Q (\mathrm{F})_Q^{12,3} = (\mathrm{F})_Q^{2,3} *_Q (\mathrm{F})_Q^{1,23}. \tag{31}$$

Proof. Write $v = 1 + v_1 + ...$, where $v_i \in \mathbf{U}_{1,f}$ has degree i and $\mathrm{F} = 1 + \mathrm{F}_1 + \mathrm{F}_2 + ...$, where $\mathrm{F}_i \in (S^{\otimes n} \boxtimes \mathbf{1})_f$ has degree i.

If we set $\mathrm{F}_1 = -f/2$, $v_1 = 0$, then (30) holds modulo terms of degree ≥ 2.

Assume that we have found $v_1, ..., v_{n-1}$ and $\mathrm{F}_1, ..., \mathrm{F}_{n-1}$ such that (30) holds modulo terms of degree $\geq n$.

Let us set $v_{<n} = 1 + v_1 + ... + v_{n-1}$, $\mathrm{F}_{<n} = 1 + \mathrm{F}_1 + ... + \mathrm{F}_{n-1}$. We then have

$$(v_{<n}^1 v_{<n}^2)^{-1} \, \mathrm{J}(r+f) v_{<n}^{12} \, \mathrm{J}(r)^{-1} = I_2(\mathrm{F}_{<n}) + \psi, \tag{32}$$

where $\psi = \psi_n + \psi_{n+1} + ...$ is an element of $\widehat{\mathbf{U}}_{2,f}$ of degree $\geq n$. Let us denote by $\mathrm{K} \in (\widehat{\mathbf{U}}_{2,f})_1^{\times}$ the l.h.s. of (32), then K satisfies

$$\mathrm{K}^{1,2} \, \mathrm{J}(r)^{1,2} \, \mathrm{K}^{12,3} (\mathrm{J}(r)^{1,2})^{-1} = \mathrm{K}^{2,3} \, \mathrm{J}(r)^{2,3} \, \mathrm{K}^{1,23} (\mathrm{J}(r)^{2,3})^{-1}.$$

This implies that

$$I_3\big((\mathrm{F}_{<n})_Q^{1,2} *_Q (\mathrm{F}_{<n})_Q^{12,3} *_Q ((\mathrm{F}_{<n})_Q^{2,3} *_Q (\mathrm{F}_{<n})_Q^{1,23})^{-1}\big) = 1 + \psi^{2,3} + \psi^{1,23} - (\psi^{1,2} + \psi^{12,3}) \tag{33}$$

modulo degree $> n$. The associated graded of I_3 is the composed map $(S^{\otimes 3}\boxtimes\mathbf{1})_f \to (\Delta(S^{\otimes 3}))_f \simeq \mathbf{U}_{n,f}$, which is injective; hence so is I_3. Therefore $(\mathrm{F}_{<n})_Q^{1,2} *_Q (\mathrm{F}_{<n})_Q^{12,3} *_Q ((\mathrm{F}_{<n})_Q^{2,3} *_Q (\mathrm{F}_{<n})_Q^{1,23})^{-1} = 1$ modulo degree $\geq n$.

Moreover, $(S^{\otimes 3}\boxtimes\mathbf{1})_f \to (\Delta(S^{\otimes 3}))_f \simeq \mathbf{U}_{3,f}$ is the linear isomorphism $(S^{\otimes 3}\boxtimes\mathbf{1})_f \xrightarrow{\sim} \mathbf{U}_{3,f}^{aaa}$, so $d(\psi_n) := \psi_n^{2,3} + \psi_n^{1,23} - (\psi_n^{1,2} + \psi_n^{12,3}) \in \mathbf{U}_{3,f}^{aaa}$.

Now $d(d(\psi_n)) = 0$ and $\mathrm{Alt}(d(\psi_n)) = 0$, so the computation of the co-Hochschild cohomology of $\mathbf{U}_{*,f}^{a\ldots a}$ in Subsection 2.5 implies that $d(\psi_n) = d(\bar{\mathrm{F}}'_n)$, where $\bar{\mathrm{F}}'_n \in \mathbf{U}_{2,f}^{aa}$. The computation of the co-Hochschild cohomology for $\mathbf{U}_{*,f}$ then implies that $\psi_n = \bar{\mathrm{F}}'_n + (v_n^{12} - v_n^1 - v_n^2) + \lambda'$, where $v_n \in \mathbf{U}_{1,f}$ and $\lambda' \in (\Delta(\wedge^2))_f$ all have degree n.

Now $(\Delta(\wedge^2))_f = (\wedge^2\boxtimes\mathbf{1})_f \oplus (\mathbf{id}\boxtimes\mathbf{id})_f \oplus (\mathbf{1}\boxtimes\wedge^2)_f$. Since $(\mathbf{1}\boxtimes\wedge^2)_f = 0$, we decompose λ' as $\lambda'' + \lambda - \lambda^{2,1}$, where $\lambda'' \in (\wedge^2\boxtimes\mathbf{1})_f$ and $\lambda \in (\mathbf{id}\boxtimes\mathbf{id})_f$. Set $\bar{\mathrm{F}}_n := \bar{\mathrm{F}}'_n + \lambda'' \in \mathbf{U}_{2,f}^{aa}$.

Then

$$\psi_n = (v_n^{12} - v_n^1 - v_n^2) + \bar{\mathrm{F}}_n + \lambda - \lambda^{2,1}.$$

Let $\mathrm{F}_n \in (S^{\otimes 2}\boxtimes\mathbf{1})_f$ be the preimage of $\bar{\mathrm{F}}_n$ under the symmetrization map $(S^{\otimes 2}\boxtimes\mathbf{1})_f \to \mathbf{U}_{2,f}^{aa}$.

Let us set $v_{\leq n} = (1+v_n)(1+v_{<n})$, $\mathrm{F}_{\leq n} = \mathrm{F}_{<n} + \mathrm{F}_n$. Then (32) is rewritten as

$$(v_{\leq n}^1 v_{\leq n}^2)^{-1}\,\mathrm{J}(r+f) v_{\leq n}^{12}\,\mathrm{J}(r)^{-1} = I_2(\mathrm{F}_{\leq n}) + \lambda - \lambda^{2,1} + \psi', \tag{34}$$

where $\psi' = \psi'_{n+1} + \ldots \in \widehat{\mathbf{U}}_{2,f}$ has degree $\geq n+1$.

As above, we denote by K' the l.h.s. of (34). We have again

$$(\mathrm{K}')^{1,2}\,\mathrm{J}(r)^{1,2}(\mathrm{K}')^{12,3}(\mathrm{J}(r)^{1,2})^{-1} = (\mathrm{K}')^{1,2}\,\mathrm{J}(r)^{2,3}(\mathrm{K}')^{1,23}(\mathrm{J}(r)^{2,3})^{-1},$$

which according to (34) can be rewritten as

$$\begin{aligned} & I_3((\mathrm{F}_{\leq n})_Q^{1,2} *_Q (\mathrm{F}_{\leq n})_Q^{12,3} *_Q ((\mathrm{F}_{\leq n})_Q^{2,3} *_Q (\mathrm{F}_{\leq n})_Q^{1,23})^{-1}) \\ & = 1 + (\psi')^{2,3} + (\psi')^{1,23} - ((\psi')^{1,2} + (\psi')^{12,3}) + [r_1^{1,2}, \lambda^{12,3} - \lambda^{3,12}]/2 - [r_1^{2,3}, \lambda^{1,23} - \lambda^{23,1}]/2 \end{aligned} \tag{35}$$

modulo terms of degree $> n+1$. Here $r_1 = \kappa_1^{\Pi}(r) \in \mathbf{U}_{2,f}^{ab}$.

As above, this equation implies that $(\mathrm{F}_{\leq n})_Q^{1,2} *_Q (\mathrm{F}_{\leq n})_Q^{12,3} *_Q ((\mathrm{F}_{\leq n})_Q^{2,3} *_Q (\mathrm{F}_{\leq n})_Q^{1,23})^{-1}$ has the form $1 + \bar{g}_{n+1} + \ldots$, where $\bar{g}_{n+1}, \ldots$ have degree $\geq n+1$. The degree $n+1$ part of (35) yields

$$g_{n+1} = (\psi'_{n+1})^{2,3} + (\psi'_{n+1})^{1,23} - ((\psi'_{n+1})^{1,2} + (\psi'_{n+1})^{12,3}) + [r_1^{1,2}, \lambda^{12,3} - \lambda^{3,12}]/2 - [r_1^{2,3}, \lambda^{1,23} - \lambda^{23,1}]/2$$

where $g_{n+1} \in \mathbf{U}_{3,f}^{aaa}$ is the image of g_{n+1} by the isomorphism $(S^{\otimes 3}\boxtimes\mathbf{1})_f \simeq \mathbf{U}_{2,f}^{aaa}$. Applying Alt to this equation, we get

$$\mathrm{Alt}([r_1^{1,2}, \lambda^{12,3} - \lambda^{3,12}] - [r_1^{2,3}, \lambda^{1,23} - \lambda^{23,1}]) \in \mathbf{U}_{3,f}^{aaa}.$$

Now the terms under Alt belong to $\mathbf{U}_{3,f}^{c_1c_2c_3}$ where $c_1c_2c_3$ is respectively aab, bba, abb, baa. These terms are antisymmetric w.r.t. the pairs of repeated indices, and $-[r_1^{2,3}, \lambda^{1,23}] = ([r_1^{1,2}, -\lambda^{3,12}])^{2,3,1}$, $[r_1^{2,3}, \lambda^{23,1}] = ([r_1^{1,2}, \lambda^{12,3}])^{2,3,1}$. Hence

$$[r_1^{1,2}, \lambda^{12,3} - \lambda^{3,12}] + \text{cyclic permutations} = 0.$$

Since the spaces $\mathbf{U}_{3,f}^{c_1c_2c_3}$ are in direct sum for distinct $c_1c_2c_3$, we get

$$[r_1^{1,2}, \lambda^{12,3}] = [r_1^{2,3}, \lambda^{1,23}] = 0.$$

We will show that the second equality implies that $\lambda = 0$. This will prove the induction step, because (34) then means that (30) holds at step $n+1$.

So it remains to prove:

Lemma 5.2. *The composition*

$$(\mathbf{id}\boxtimes\mathbf{id})_f \hookrightarrow \mathbf{U}^{ab}_{2,f} \to \mathbf{U}_{3,f} \tag{36}$$

is injective, where the first map is $(\mathbf{id}\boxtimes\mathbf{id})_f \simeq \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, \mathbf{id}\underline{\boxtimes}\mathbf{id}) \hookrightarrow \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}S) \simeq \mathbf{U}^{ab}_{2,f}$ *and the second map is* $\mathbf{U}^{ab}_{2,f} \ni \lambda \mapsto [r_1^{2,3}, \lambda^{1,23}] \in \mathbf{U}_{3,f}$.

Proof of Lemma. It follows from Subsection 1.22 that (36) coincides with the composition $(\mathbf{id}\boxtimes\mathbf{id})_f \to (\mathbf{id}\boxtimes\wedge^2)_f \hookrightarrow (\mathbf{id}\boxtimes\mathbf{id}^{\otimes 2})_f \hookrightarrow \mathbf{U}^{abb}_{3,f} \hookrightarrow \mathbf{U}_{3,f}$, where the first map is

$$(\mathbf{id}\boxtimes\mathbf{id})_f \simeq \mathrm{LBA}_f(\mathbf{id},\mathbf{id}) \stackrel{-\circ\mu}{\to} \mathrm{LBA}_f(\wedge^2,\mathbf{id}) \simeq (\mathbf{id}\boxtimes\wedge^2)_f,$$

the second and the fourth maps are the natural injections, and the third map is the injection

$$(\mathbf{id}\boxtimes\mathbf{id}^{\otimes 2})_f \simeq \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (\mathbf{id}\underline{\boxtimes}\mathbf{1})\otimes(\mathbf{1}\underline{\boxtimes}\mathbf{id})^{\otimes 2}) \hookrightarrow \Pi_f(\mathbf{1}\underline{\boxtimes}\mathbf{1}, (S\underline{\boxtimes}\mathbf{1})\otimes(\mathbf{1}\underline{\boxtimes}S)^{\otimes 2}) \simeq \mathbf{U}^{abb}_{3,f}.$$

It follows from Proposition 3.3 that the first map is also injective. Therefore the map $(\mathbf{id}\boxtimes\mathbf{id})_f \to \mathbf{U}_{3,f}$ given by (36) is injective. This proves the Lemma. □

This ends the proof of the first part of Theorem 5.1. Equation (31) is then obtained by taking the limit $n\to\infty$ in (33). This proves Theorem 5.1. □

We prove that pairs (v,F) are unique up to gauge (this fact will not be used in the sequel).

Lemma 5.3. *The set of pairs* (v,F) *as in Theorem 5.1 is a torsor under the action of* $((\widehat{S\boxtimes\mathbf{1}})_f)_1^\times$*: an element* $g\in((\widehat{S\boxtimes\mathbf{1}})_f)_1^\times$ *transforms* (v,F) *into* $(vI_1(g), ((g)^1_Q *_Q (g)^2_Q)^{-1} *_Q \mathrm{F} *_Q (g)^{12}_Q)$.

Proof. Since $I_2(g^{12}_Q) = \mathrm{J}(r)I_1(g)^{12}\,\mathrm{J}(r)^{-1}$, the pair $(vI_1(g), ((g)^1_Q *_Q (g)^2_Q)^{-1} *_Q \mathrm{F} *_Q (g)^{12}_Q)$ is also a solution of the equation of Theorem 5.1. Conversely, let (v_1,F_1) and (v_2,F_2) be solutions of this equation. Then $v_1^1 v_1^2 I_2(\mathrm{F}_1)\,\mathrm{J}(r)(v_1^{12})^{-1} = v_2^1 v_2^2 I_2(\mathrm{F}_2)\,\mathrm{J}(r)(v_2^{12})^{-1}$. Let n be the smallest index such that the degree n components of (v_1,F_1) and (v_2,F_2) are different. We denote with an additional index n these components. Then we have

$$(v_{2,n}-v_{1,n})^{12} - (v_{2,n}-v_{1,n})^1 - (v_{2,n}-v_{1,n})^2 = \mathrm{sym}_2(\mathrm{F}_{2,n}-\mathrm{F}_{1,n}),$$

where $\mathrm{sym}_2 : (S^{\otimes 2}\boxtimes\mathbf{1})_f \to (\Delta(S^{\otimes 2}))_f \simeq \mathbf{U}_{2,f}$ is the canonical injection. So $d(v_{2,n}-v_{1,n}) \in \mathbf{U}^{aa}_{2,f}$. As above, we obtain the existence of $w\in\mathbf{U}^a_{1,f}$ of degree n, such that $d(v_{2,n}-v_{1,n}) = d(w)$. Therefore $v_{2,n}-v_{1,n}-w \in (\Delta(\mathbf{id}))_f = (\mathbf{id}\boxtimes\mathbf{1})_f \oplus (\mathbf{1}\boxtimes\mathbf{id})_f$. Now $(\mathbf{1}\boxtimes\mathbf{id})_f = 0$, so $v_{2,n}-v_{1,n}-w \in (\mathbf{id}\boxtimes\mathbf{1})_f \subset \mathbf{U}^a_{1,f}$. Therefore $w' := v_{2,n}-v_{1,n} \in \mathbf{U}^a_{1,f}$. Replacing (v_2,F_2) by $(v_2 I_1(1-w'), ((1-w')^1_Q *_Q (1-w')^2_Q)^{-1} *_Q \mathrm{F}_2 *_Q (1-w')^{12}_Q)$, we obtain a solution equal to (v_1,F_1) up to degree n. Proceeding inductively, we see that (v_1,F_1) and (v_2,F_2) are related by the action of an element of $((\widehat{S\boxtimes\mathbf{1}})_f)_1^\times$. □

5.6. **Compatibility of quantization functors with twists.** Let $\mathbf{P}$ be a prop and $\overline{Q}$: Bialg $\to S(\mathbf{P})$ be a prop morphism. Using $\overline{Q}$, we equip the collection of all $\mathbf{P}(\mathbf{1}, S^{\otimes n})$ with the structure of a $\mathcal{Y}$-algebra (see Subsection 5.3) with unit $\overline{Q}(\eta^{\otimes n})$.

We define a twist of $\overline{Q}$ to be an element F of $\mathbf{P}(\mathbf{1}, S^{\otimes 2})^\times$, such that the relations

$$(\mathrm{F})^{1,2}_{\overline{Q}} *_{\overline{Q}} (\mathrm{F})^{12,3}_{\overline{Q}} = (\mathrm{F})^{2,3}_{\overline{Q}} *_{\overline{Q}} (\mathrm{F})^{1,23}_{\overline{Q}}, \quad (\mathrm{F})^{\emptyset,1}_{\overline{Q}} = (\mathrm{F})^{1,\emptyset}_{\overline{Q}} = \overline{Q}(\eta)$$

hold in $\mathbf{P}(\mathbf{1}, S^{\otimes 3})$ (here we use the $\mathcal{Y}$-algebra structure on $\mathbf{P}(\mathbf{1}, S^{\otimes n})$ given by $\overline{Q}$). Then get a new prop morphism ${}^{\mathrm{F}}\overline{Q}$: Bialg $\to S(\mathbf{P})$, defined by ${}^{\mathrm{F}}\overline{Q}(m) = \overline{Q}(m)$, ${}^{\mathrm{F}}\overline{Q}(\Delta) = \underline{\mathrm{Ad}}(\mathrm{F})\circ\overline{Q}(\Delta)$, ${}^{\mathrm{F}}\overline{Q}(\varepsilon) = \overline{Q}(\varepsilon)$, ${}^{\mathrm{F}}\overline{Q}(\eta) = \overline{Q}(\eta)$. Here $\underline{\mathrm{Ad}}(\mathrm{F}) \in S(\mathbf{P})(\mathbf{id}^{\otimes 2}, \mathbf{id}^{\otimes 2})$ is given by

$$\underline{\mathrm{Ad}}(\mathrm{F}) = \overline{Q}(m^{(2)}\otimes m^{(2)})\circ(142536)\circ(\mathrm{F}\otimes\mathrm{id}_{S^{\otimes 2}}\otimes\mathrm{F}^{-1}) \in \mathbf{P}(S^{\otimes 2}, S^{\otimes 2});$$

here $m^{(2)} = m\circ(m\otimes\mathrm{id}_{\mathbf{id}}) \in \mathrm{Bialg}(T_3, \mathbf{id})$.

We say that the prop morphisms $\overline{Q},\overline{Q}'$: Bialg $\to S(\mathbf{P})$ are equivalent if $\overline{Q}' = \theta(\xi)\circ\overline{Q}$, where $\xi\in S(\mathbf{P})(\mathbf{id},\mathbf{id})^\times$ and $\theta(\xi)$ is the corresponding inner automorphism of $S(\mathbf{P})$.

Theorem 5.4. *Let Q : Bialg $\to S(\mathbf{LBA})$ be an Etingof-Kazhdan quantization functor. Then $S(\kappa_i)\circ Q$: Bialg $\to S(\mathbf{LBA}_f)$ $(i=1,2)$ are prop morphisms. There exists $\mathrm{i}\in S(\mathbf{LBA}_f)(\mathbf{id},\mathbf{id})^\times$, such that $\kappa_0(\mathrm{i}) = S(\mathbf{LBA})(\mathrm{id}_{\mathbf{id}})$, and a twist F of $S(\kappa_1)\circ Q$, such that*

$$S(\kappa_2)\circ Q = \theta(\mathrm{i})\circ{}^{\mathrm{F}}(S(\kappa_1)\circ Q).$$

Proof. We will construct i, such that

$$\kappa_2^\Pi(m_a) = \mathrm{i}\circ\kappa_1^\Pi(m_a)\circ(\mathrm{i}^{\boxtimes 2})^{-1},\quad \kappa_2^\Pi(\Delta_a) = \mathrm{i}^{\boxtimes 2}\circ\underline{\mathrm{Ad}}(\mathrm{F})\circ\kappa_1^\Pi(\Delta_a)\circ\mathrm{i}^{-1},$$

where as before

$$\underline{\mathrm{Ad}}(\mathrm{F}) = (m_a^{(2)}\boxtimes m_a^{(2)})\circ(142536)\circ\big(\mathrm{F}\boxtimes\mathrm{id}_{(S\underline{\boxtimes}\mathbf{1})^{\boxtimes 2}}\boxtimes\mathrm{F}^{-1}\big)\in\mathbf{\Pi}_f((S\underline{\boxtimes}\mathbf{1})^{\otimes 2},(S\underline{\boxtimes}\mathbf{1})^{\otimes 2})^\times.$$

Let us relate the $\kappa_i^\Pi(m_\Pi)$, $i=1,2$. (20) implies that

$$\kappa_2^\Pi\big(\mathrm{Ad}(\mathrm{J})\big) = \Xi_f^{\boxtimes 2}\circ\mathrm{Ad}(v)^{\otimes 2}\circ\kappa_1^\Pi(\mathrm{Ad}(\mathrm{R}_+^{\boxtimes 2}\circ\mathrm{F})\circ\mathrm{Ad}(\mathrm{J}))\circ\mathrm{Ad}(v^{12})^{-1}\circ(\Xi_f^{-1})^{\boxtimes 2},$$

and therefore (21) implies that

$$\kappa_2^\Pi(\Delta_\Pi) = \Xi_f^{\boxtimes 2}\circ\mathrm{Ad}(v)^{\boxtimes 2}\circ\kappa_1^\Pi(\mathrm{Ad}(\mathrm{R}_+^{\boxtimes 2}\circ\mathrm{F})\circ\Delta_\Pi)\circ\mathrm{Ad}(v)^{-1}\circ\Xi_f^{-1}.$$

Now $\mathrm{R}(r+f) = (m_\Pi^{(2)}\boxtimes m_\Pi^{(2)})\circ(142536)\circ\Big((\mathrm{R}_+^{\boxtimes 2}\circ\mathrm{F}^{2,1})\boxtimes\mathrm{R}(r)\boxtimes(\mathrm{R}_+^{\boxtimes 2}\circ\mathrm{F}^{-1})\Big)$, where $\mathrm{F}\in\mathbf{\Pi}_f(\mathbf{1}\underline{\boxtimes}\mathbf{1},(S\underline{\boxtimes}\mathbf{1})^{\otimes 2})$. For $X\in\mathbf{\Pi}_f(\mathbf{1}\underline{\boxtimes}\mathbf{1},(S\underline{\boxtimes}\mathbf{1})^{\otimes 2})$, set $\underline{X} := (\mathrm{can}^*_{\mathbf{1}\underline{\boxtimes}S}\boxtimes\mathrm{id}_{S\underline{\boxtimes}\mathbf{1}})\circ(\mathrm{id}_{\mathbf{1}\underline{\boxtimes}S}\boxtimes X)$.

Then

$$(\kappa_2^\Pi(\mathrm{R}_+)\boxtimes\kappa_2^\Pi(\mathrm{R}_-))\circ\mathrm{can}_{S\underline{\boxtimes}\mathbf{1}} = \kappa_2^\Pi(\mathrm{R}) = \Xi_f^{\boxtimes 2}\circ\mathrm{R}(r+f) = \Big(\big(\Xi_f\circ\mathrm{Ad}(v)\circ\kappa_1^\Pi(\mathrm{R}_+)\big)$$
$$\boxtimes\big(\Xi_f\circ\mathrm{Ad}(v)\circ\kappa_1^\Pi\big(m_\Pi^{(2)*}\circ(\mathrm{R}_+\boxtimes\mathrm{R}_-\boxtimes\mathrm{R}_+)\big)\circ\big(\underline{\mathrm{F}}^{2,1}\boxtimes\mathrm{id}_{\mathbf{1}\underline{\boxtimes}S}\boxtimes\underline{\mathrm{F}}^{-1}\big)\big)\Big)\circ\mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}.$$

It follows that for some $\mathrm{i}\in\mathbf{\Pi}_f(S\underline{\boxtimes}\mathbf{1},S\underline{\boxtimes}\mathbf{1})^\times$, we have

$$\kappa_2^\Pi(\mathrm{R}_+) = \Xi_f\circ\mathrm{Ad}(v)\circ\kappa_1^\Pi(\mathrm{R}_+)\circ\mathrm{i}^{-1},$$

therefore $\kappa_2^\Pi(\mathrm{R}_+^{(-1)}) = \mathrm{i}\circ\kappa_1^\Pi(\mathrm{R}_+^{(-1)})\circ\mathrm{Ad}(v^{-1})\circ\Xi_f^{-1}$.

Now

$$\kappa_2^\Pi(m_a) = \kappa_2^\Pi(\mathrm{R}_+^{(-1)})\circ\kappa_2^\Pi(m_\Pi)\circ\kappa_2^\Pi(\mathrm{R}_+)^{\boxtimes 2} = \mathrm{i}\circ\kappa_1^\Pi(\mathrm{R}_+^{(-1)})\circ\kappa_1^\Pi(m_\Pi)\circ\kappa_1^\Pi(\mathrm{R}_+^{\boxtimes 2})\circ(\mathrm{i}^{-1})^{\boxtimes 2}$$
$$= \mathrm{i}\circ\kappa_1^\Pi(m_a)\circ(\mathrm{i}^{-1})^{\boxtimes 2},$$

and

$$\kappa_2^\Pi(\Delta_a) = \kappa_2^\Pi(\mathrm{R}_+^{(-1)})^{\boxtimes 2}\circ\kappa_2^\Pi(\Delta_\Pi)\circ\kappa_2^\Pi(\mathrm{R}_+) = \mathrm{i}^{\boxtimes 2}\circ\kappa_1^\Pi(\mathrm{R}_+^{(-1)})^{\boxtimes 2}\circ\kappa_1^\Pi(\mathrm{Ad}(\mathrm{R}_+^{\boxtimes 2}\circ\mathrm{F})\circ\Delta_\Pi)\circ\kappa_1^\Pi(\mathrm{R}_+)\circ\mathrm{i}^{-1}.$$

We first prove that

$$(\kappa_1^\Pi(\mathrm{R}_+^{(-1)}))^{\boxtimes 2}\circ\mathrm{Ad}(\kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\mathrm{F})\circ\kappa_1^\Pi(\Delta_\Pi\circ\mathrm{R}_+) = \underline{\mathrm{Ad}}(\mathrm{F})\circ\kappa_1^\Pi(\Delta_a). \tag{37}$$

One checks that $\mathrm{Ad}(\kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\mathrm{F})\circ\kappa_1^\Pi(\Delta_\Pi\circ\mathrm{R}_+) = \kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\underline{\mathrm{Ad}}(\mathrm{F})$. We have

$$\Delta_\Pi\circ\mathrm{R}_+ = \Delta_\Pi\circ\mathrm{R}_+\circ\mathrm{R}_-^*\circ(\mathrm{R}_-^{(-1)})^* = \Delta_\Pi\circ\ell\circ(\mathrm{R}_-^{(-1)})^*$$
$$= \ell^{\boxtimes 2}\circ\overline{\Delta}_\Pi\circ(\mathrm{R}_-^{(-1)})^* = \mathrm{R}_+^{\boxtimes 2}\circ(\mathrm{R}_-^*)^{\boxtimes 2}\circ\overline{\Delta}_\Pi\circ(\mathrm{R}_-^{(-1)})^* = \mathrm{R}_+^{\boxtimes 2}\circ\Delta_a.$$

Applying κ_1^Π and composing from the left with $\mathrm{Ad}(\kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\mathrm{F})$, we get $\mathrm{Ad}(\kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\mathrm{F})\circ\kappa_1^\Pi(\Delta_\Pi\circ\mathrm{R}_+) = \kappa_1^\Pi(\mathrm{R}_+)^{\boxtimes 2}\circ\underline{\mathrm{Ad}}(\mathrm{F})\circ\kappa_1^\Pi(\Delta_a)$. Composing from the left with $\kappa_1^\Pi(\mathrm{R}_+^{(-1)})^{\boxtimes 2}$, we get (37).

It follows that

$$\kappa_2^\Pi(\Delta_a) = \mathrm{i}^{\boxtimes 2}\circ\underline{\mathrm{Ad}}(\mathrm{F})\circ\kappa_1^\Pi(\Delta_a)\circ\mathrm{i}^{-1},$$

as wanted. □

6. Quantization of coboundary Lie bialgebras

6.1. **Compatibility with coopposite.** Let Φ be an associator. Then $\Phi' := \Phi(-A, -B)$ is also an associator. Let Q, Q' be the Etingof-Kazhdan quantization functors corresponding to Φ, Φ'.

Recall that $\tau_{\mathrm{LBA}} \in \mathrm{Aut}(\mathrm{LBA})$ is defined by $\mu \mapsto \mu$, $\delta \mapsto -\delta$ and let $\tau_{\mathrm{Bialg}} \in \mathrm{Aut}(\mathrm{Bialg})$ be defined by $m \mapsto m$, $\Delta \mapsto (21) \circ \Delta$.

Proposition 6.1. *There exists $\xi_\tau \in S(\mathbf{LBA})(\mathbf{id}, \mathbf{id})^\times$, with $\xi_\tau = \mathrm{id}_{\mathbf{id}}$ + terms of positive degree in both μ and δ, such that*

$$Q' \circ \tau_{\mathrm{Bialg}} = \theta(\xi_\tau) \circ S(\tau_{\mathrm{LBA}}) \circ Q.$$

Proof. This means that

$$Q'(m) = \xi_\tau^{\boxtimes 2} \circ S(\tau_{\mathrm{LBA}})(Q(m)) \circ \xi_\tau^{-1}, \quad Q'(\Delta) \circ (21) = \xi_\tau \circ S(\tau_{\mathrm{LBA}})(Q(\Delta)) \circ (\xi_\tau^{-1})^{\boxtimes 2}.$$

We will therefore construct $\tilde{\xi}_\tau \in \mathbf{\Pi}(S\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}\mathbf{1})^\times$, such that

$$m'_a = \tilde{\xi}_\tau^{\boxtimes 2} \circ \tau_\Pi(m_a) \circ \tilde{\xi}_\tau^{-1}, \quad \Delta'_a \circ (21) = \tilde{\xi}_\tau \circ \tau_\Pi(\Delta_a) \circ (\tilde{\xi}_\tau^{\boxtimes 2})^{-1},$$

where m'_a, Δ'_a are the analogues of m_a, Δ_a for Φ'.

Lemma 6.2. $\tau_\Pi(\mathrm{Ad}(\mathrm{J})) = (\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes 2} \circ \mathrm{Ad}(\mathrm{J}(-r)) \circ ((\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes 2})^{-1}$.

This follows from Lemma 1.19.

Lemma 6.3. *Let J' be the analogue of J for Φ'. There exists $u \in \widehat{\mathbf{U}}_1$ of the form $u = 1$ + terms of degree ≥ 1, such that*

$$(\mathrm{J}')^{2,1} = u^1 u^2\, \mathrm{J}(-r) (u^{12})^{-1}. \tag{38}$$

Proof of Lemma. We have

$$\mathrm{J}(-r)^{1,2}\, \mathrm{J}(-r)^{12,3} = \mathrm{J}(-r)^{2,3}\, \mathrm{J}(-r)^{1,23} \Phi(-t_{12}, -t_{23}) = \mathrm{J}(-r)^{2,3}\, \mathrm{J}(-r)^{1,23} \Phi'$$

(equality in $\widehat{\mathbf{U}}_2$, where we use the $\mathcal{X}$-algebra structure on $\widehat{\mathbf{U}}_n$). Let us set $u_0 = 1$ + class of $a_1 b_1/2$ and $\overline{\mathrm{J}} := u_0^1 u_0^2\, \mathrm{J}(-r)^{2,1} (u_0^{12})^{-1}$. Then $\overline{\mathrm{J}}$ satisfies $\overline{\mathrm{J}}^{1,2} \overline{\mathrm{J}}^{12,3} = \overline{\mathrm{J}}^{2,3} \overline{\mathrm{J}}^{1,23} \Phi'$ (since $(\Phi')^{3,2,1} = (\Phi')^{-1}$) and $\overline{\mathrm{J}} = 1 - r/2$ + terms of degree > 1, and J' satisfies the same conditions. According to [Enr3], this implies the existence of $u_1 \in \widehat{\mathbf{U}}_1$ of the form $u_1 = 1$ + terms of degree > 1, such that $\mathrm{J}' = u_1^1 u_1^2 \overline{\mathrm{J}} (u_1^{12})^{-1}$, so if we set $u = u_1 u_0 \in \widehat{\mathbf{U}}_1$, then u has the form $u = 1$ + class of $a_1 b_1/2$ + terms of degree > 1, and satisfies (38). □

Lemma 6.4. *We have* $\tau_\Pi(\Delta_\Pi) = (\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes 2} \circ \mathrm{Ad}(u^{-1})^{\boxtimes 2} \circ \Big((21) \circ \Delta'_\Pi\Big) \circ \mathrm{Ad}(u) \circ (\mathrm{id}_S \underline{\boxtimes} \omega_S)^{-1}$.

This follows from Lemmas 1.19 and 6.2.

Lemma 6.5. $\tau_\Pi(\mathrm{R}) = (\mathrm{id}_S \underline{\boxtimes} \omega_S)^{\boxtimes 2} \circ \mathrm{Ad}(u^{-1})^{\boxtimes 2} \circ (\mathrm{R}')^{-1}$.

Proof of Lemma. We have $\mathrm{R} = \sum_{n\geq 0}(n!)^{-1}(m_\Pi^{(n+1)} \boxtimes m_\Pi^{(n+1)}) \circ (1, n+3, 2, n+4, ...) \circ \big(\mathrm{J}^{2,1} \boxtimes t^{\boxtimes n} \circ \mathrm{J}^{-1}\big)$, so

$$\tau_\Pi(\mathrm{R}) = \sum_{n\geq 0}(n!)^{-1}(\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\boxtimes 2} \circ (m_\Pi^{(n+1)} \boxtimes m_\Pi^{(n+1)}) \circ (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\boxtimes 2(n+2)})^{-1}$$
$$\circ \big(\tau_\Pi(\mathrm{J}^{2,1}) \boxtimes \tau_\Pi(t)^{\boxtimes n} \boxtimes \tau_\Pi(\mathrm{J}^{-1})\big)$$
$$= \sum_{n\geq 0}(n!)^{-1}(\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\boxtimes 2} \circ (m_\Pi^{(n+1)} \boxtimes m_\Pi^{(n+1)}) \circ \big(\mathrm{J}(-r)^{2,1} \boxtimes (-t)^{\boxtimes n} \boxtimes \mathrm{J}(-r)^{-1}\big)$$
$$= (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\boxtimes 2} \circ \big(\mathrm{J}(-r)^{2,1} e^{-t/2}\, \mathrm{J}(-r)\big) = (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{\boxtimes 2} \circ \mathrm{R}(-r).$$

Now $\mathrm{R} = (u^1u^2)(\mathrm{R}')^{-1}(u^1u^2)^{-1}$, whence the result. □

Lemma 6.6. *There exists $\sigma \in \mathbf{\Pi}(\mathbf{1}\underline{\boxtimes}S, \mathbf{1}\underline{\boxtimes}S)^\times$, such that*

$$\mathrm{R}^{-1} = \Big(\mathrm{R}_+ \boxtimes (\mathrm{R}_- \circ \sigma)\Big) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}\,.$$

Proof. Set can $:= \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}$ and $\mathrm{can}_+ := \sum_{i\geq 1} \mathrm{can}_{S^i\underline{\boxtimes}\mathbf{1}}$. Set $m_b := \Delta_a^*$. Then $m_\Pi \circ \mathrm{R}_-^{\boxtimes 2} = \mathrm{R}_- \circ m_b$. The series $\mathrm{can}' := \mathrm{can}_{\mathbf{1}\underline{\boxtimes}\mathbf{1}} + \sum_{i>0}(m_a^{(i)} \boxtimes m_a^{(i)}) \circ (1, i+1, 2, i+2, ...) \circ (-\mathrm{can}_+)^{\boxtimes i}$ is convergent and has the form $(\mathrm{id}_{S\underline{\boxtimes}\mathbf{1}} \boxtimes \sigma) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}$ for a suitable invertible σ. We then have $(m_a \boxtimes m_b) \circ (32) \circ (\mathrm{can}_{S\underline{\boxtimes}\mathbf{1}} \boxtimes \mathrm{can}') = \mathrm{can}_{\mathbf{1}\underline{\boxtimes}\mathbf{1}}$.

It follows that $\mathrm{R}^{-1} = (\mathrm{R}_+ \boxtimes \mathrm{R}_-) \circ \mathrm{can}' = \Big(\mathrm{R}_+ \boxtimes (\mathrm{R}_- \circ \sigma)\Big) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}$. □

End of proof of Proposition 6.1. The above lemmas imply

$$\tau_\Pi(\mathrm{R}) = \Big((\mathrm{id}_S \boxtimes \omega_S) \circ \mathrm{Ad}(u^{-1})\Big)^{\boxtimes 2} \circ \Big(\mathrm{R}'_+ \boxtimes (\mathrm{R}'_- \circ \sigma')\Big) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}},$$

where σ' is the analogue of σ for Φ'. Since $\tau_\Pi(\mathrm{R}) = \Big(\tau_\Pi(\mathrm{R}_+) \boxtimes \tau_\Pi(\mathrm{R}_-)\Big) \circ \mathrm{can}_{S\underline{\boxtimes}\mathbf{1}}$, there exists $\xi_\tau \in \mathbf{\Pi}(S\underline{\boxtimes}\mathbf{1}, S\underline{\boxtimes}\mathbf{1})^\times$, such that

$$\tau_\Pi(\mathrm{R}_+) = (\mathrm{id}_S \underline{\boxtimes}\omega_S) \circ \mathrm{Ad}(u)^{-1} \circ \mathrm{R}'_+ \circ \xi_\tau.$$

It follows that

$$\tau_\Pi(\mathrm{R}_+^{(-1)}) = \xi_\tau^{-1} \circ \mathrm{R}_+'^{(-1)} \circ \mathrm{Ad}(u) \circ (\mathrm{id}_S \underline{\boxtimes}\omega_S)^{-1}.$$

Then

$$\tau_\Pi(m_a) = \tau_\Pi(\mathrm{R}_+^{(-1)} \circ m_\Pi \circ \mathrm{R}_+^{\boxtimes 2}) = \xi_\tau^{-1} \circ \mathrm{R}_+'^{(-1)} \circ \mathrm{Ad}(u) \circ m_\Pi \circ (\mathrm{Ad}(u)^{-1})^{\boxtimes 2} \circ (\mathrm{R}'_+)^{\boxtimes 2} \circ \xi_\tau^{\boxtimes 2}$$
$$= \xi_\tau^{-1} \circ m'_a \circ \xi_\tau^{\boxtimes 2}$$

and

$$\tau_\Pi(\Delta_a) = \tau_\Pi\Big((\mathrm{R}_+^{(-1)})^{\boxtimes 2} \circ \Delta_\Pi \circ \mathrm{R}_+\Big) = \Big(\xi_\tau^{-1} \circ \mathrm{R}_+'^{(-1)}\Big)^{\boxtimes 2} \circ (21) \circ \Delta'_\Pi \circ (\mathrm{R}'_+)^{\boxtimes 2} \circ \xi_\tau$$
$$= (\xi_\tau^{-1})^{\boxtimes 2} \circ \Delta'_a \circ \xi_\tau.$$

Moreover, the image $(\xi_\tau)_{|\mu=\delta=0} \in S(\mathbf{Sch})(\mathbf{id}, \mathbf{id})$ of ξ_τ by the morphism $\mathbf{LBA} \to \mathbf{Sch}$, $\mu, \delta \mapsto 0$ is equal to $\mathrm{id}_{\mathbf{id}}$. Set $\xi' := (\xi_\tau)_{|\delta=0}$, then $\xi' \in \mathbf{LA}(S, S)$. We have $\mathrm{LA}(S^p, S^q) = 0$ unless $p = q$, and $\mathrm{LA}(S^p, S^p) = \mathbf{k}\, \mathrm{id}_{S^p}$. So $\xi' = \mathrm{id}_S$. Now $\xi_\tau = \xi' +$ terms of positive degree in δ, so $\xi_\tau = \mathrm{id}_S +$ terms of positive degree in δ. In the same way, $\xi_\tau = \mathrm{id}_S +$ terms of positive degree in μ. So $\xi_\tau = \mathrm{id}_S +$ terms of positive degree in both δ and μ.

This ends the proof of Proposition 6.1. □

Remark 6.7. We take this opportunity to correct a mistake in Theorem 2.1 in [Enr3]. Let $\mathrm{J} = 1 - r/2 + ...$ be a solution of (24). Then the set of solutions of (24) of the form 1+ terms of degree ≥ 1 consists in the disjoint union of *two* gauge orbits (and not one), that of J and that of $\mathrm{J}^{2,1}$. The degree one term of the solution has the form $\alpha r + \beta r^{2,1}$, where $\alpha - \beta = \pm 1/2$; the solution is in the gauge class of J (resp., $\mathrm{J}^{2,1}$) iff $\alpha - \beta = -1/2$ (resp., $1/2$). This follows from a more careful analysis in degree one in the proof of Theorem 2.1 in [Enr3].

6.2. **Quantization functors for coboundary Lie bialgebras.** A quantization functor of coboundary Lie bialgebras is a prop morphism $\overline{Q} : \mathrm{COB} \to S(\mathbf{Cob})$, such that:

(a) the composed morphism $\mathrm{Bialg} \to \mathrm{COB} \xrightarrow{\overline{Q}} S(\mathbf{Cob}) \xrightarrow{\mu, r \mapsto 0} S(\mathbf{Sch})$ is the propic version of the bialgebra structure on the symmetric algebras,

(b) $\overline{Q}(R) = \mathrm{inj}_0^{\otimes 2}$ + terms of degree ≥ 1 in ρ, and $\overline{Q}(R) - (21) \circ \overline{Q}(R) = \mathrm{inj}_1^{\otimes 2} \circ \rho$ + terms of degree ≥ 2 in ρ, where $\mathrm{inj}_0 \in \mathbf{Sch}(\mathbf{1}, S)$ and $\mathrm{inj}_1 \in \mathbf{Sch}(\mathbf{id}, S)$ are the canonical injection maps (recall that Cob has a grading where μ has degree 0 and ρ has degree 1).

As in the case of quantization functors of Lie bialgebras, $\overline{Q}$ necessarily satisfies $\overline{Q}(\eta) = \mathrm{inj}_0$, $\overline{Q}(\varepsilon) = \mathrm{pr}_0$. As we explained, each such morphism $\overline{Q}$ yields a solution of the quantization problem of coboundary Lie bialgebras.

6.3. **Construction of quantization functors of coboundary Lie bialgebras.**

Theorem 6.8. *Any even associator defined over* $\mathbf{k}$ *gives rise to a quantization functor of coboundary Lie bialgebras.*

Remark 6.9. In [BN], the existence of rational even associators is proved. This implies the existence of quantization functors of coboundary Lie bialgebras over any field $\mathbf{k}$ of characteristic 0.

Proof. There is a unique automorphism τ_{Cob} of Cob, defined by $\mu \mapsto \mu$ and $\rho \mapsto -\rho$. Then the following diagrams of prop morphisms commute

$$\begin{array}{ccc} \mathrm{LBA} & \overset{\kappa\circ\kappa_1}{\to} & \mathrm{Cob} \\ {\scriptstyle \tau_{\mathrm{LBA}}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathrm{Cob}}} \\ \mathrm{LBA} & \overset{\kappa\circ\kappa_1}{\to} & \mathrm{Cob} \end{array} \quad \text{and} \quad \begin{array}{cccc} & {\scriptstyle \kappa_1}\nearrow & \mathrm{LBA}_f \overset{\kappa}{\to} & \mathrm{Cob} \\ \mathrm{LBA} & & & \downarrow{\scriptstyle \tau_{\mathrm{Cob}}} \\ & {\scriptstyle \kappa_2}\searrow & \mathrm{LBA}_f \overset{\kappa}{\to} & \mathrm{Cob} \end{array} \tag{39}$$

Let $Q : \mathrm{Bialg} \to S(\mathbf{LBA})$ be a quantization functor corresponding to an even associator. Then $\overline{Q} := S(\kappa \circ \kappa_1) \circ Q : \mathrm{Bialg} \to S(\mathbf{Cob})$ is a prop morphism. We have

$$\begin{aligned} S(\tau_{\mathrm{Cob}}) \circ S(\kappa \circ \kappa_1) \circ Q &= S(\kappa \circ \kappa_1) \circ S(\tau_{\mathrm{LBA}}) \circ Q = S(\kappa \circ \kappa_1) \circ \theta(\xi_\tau^{-1}) \circ Q \circ \tau_{\mathrm{Bialg}} \\ &= \theta(S(\kappa \circ \kappa_1)(\xi_\tau^{-1})) \circ S(\kappa \circ \kappa_1) \circ Q \circ \tau_{\mathrm{Bialg}} \end{aligned}$$

(the first equality uses the first diagram of (39), and the second equality uses Proposition 6.1), so

$$S(\tau_{\mathrm{Cob}}) \circ \overline{Q} = \theta(\xi'_\tau) \circ \overline{Q} \circ \tau_{\mathrm{Bialg}},$$

where $\xi_\tau = S(\kappa \circ \kappa_1)(\xi_\tau^{-1})$.

On the other hand, there exists $\mathrm{F} \in \mathbf{LBA}_f(\mathbf{1}, S^{\otimes 2})^\times$ and $\mathrm{i} \in S(\mathbf{LBA}_f)(\mathbf{id}, \mathbf{id})^\times$, such that $S(\kappa_2) \circ Q = \theta(\mathrm{i}) \circ {}^{\mathrm{F}}(S(\kappa_1) \circ Q)$. Composing this equality with $S(\kappa)$, we get: $S(\kappa \circ \kappa_2) \circ Q = \theta(S(\kappa)(\mathrm{i})) \circ {}^{S(\kappa)(\mathrm{F})}(\overline{Q})$. Now $S(\kappa \circ \kappa_2) \circ Q = S(\tau_{\mathrm{Cob}}) \circ \overline{Q}$ (using the second diagram in (39)), so

$$S(\tau_{\mathrm{Cob}}) \circ \overline{Q} = \theta(S(\kappa)(\mathrm{i})) \circ {}^{S(\kappa)(\mathrm{F})}(\overline{Q}).$$

We therefore get $\theta(\xi'_\tau) \circ \overline{Q} \circ \tau_{\mathrm{Bialg}} = \theta(S(\kappa)(\mathrm{i})) \circ {}^{S^{\otimes 2}(\kappa)(\mathrm{F})}\overline{Q}$, so

$${}^{\mathrm{F}'}\overline{Q} = \theta(\xi'') \circ \overline{Q} \circ \tau_{\mathrm{Bialg}}, \tag{40}$$

where $\xi'' = S(\kappa)(\mathrm{i})^{-1} \circ \xi'_\tau$, and $\mathrm{F}' = S^{\otimes 2}(\kappa)(\mathrm{F})$. Here $\xi'' \in S(\mathbf{Cob})(\mathbf{id}, \mathbf{id})^\times$ has the form id_S + terms of positive degree in ρ, and $\mathrm{F}' \in \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$ satisfies

$$(\mathrm{F}')^{1,2}_{\overline{Q}} *_{\overline{Q}} (\mathrm{F}')^{12,3}_{\overline{Q}} = (\mathrm{F}')^{2,3}_{\overline{Q}} *_{\overline{Q}} (\mathrm{F}')^{1,23}_{\overline{Q}}, \quad (\mathrm{F}')^{\emptyset,1}_{\overline{Q}} = (\mathrm{F}')^{1,\emptyset}_{\overline{Q}} = \mathrm{inj}_0, \tag{41}$$

and

$$\mathrm{F}' = \mathrm{inj}_0^{\otimes 2} + \rho + \text{terms of degree } \geq 2 \text{ in } \rho. \tag{42}$$

We will prove:

Proposition 6.10. *There exists* $\mathrm{G} \in \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$*, satisfying (41) (where* $\overline{Q}$ *is also used), (42), and*

$$\mathrm{G} *_{\overline{Q}} \mathrm{G}^{2,1} = \mathrm{G}^{2,1} *_{\overline{Q}} \mathrm{G} = \mathrm{inj}_0^{\boxtimes 2}, \quad (21) \circ \overline{Q}(\Delta) = \underline{\mathrm{Ad}}(\mathrm{G}) \circ \overline{Q}(\Delta)$$

(recall that the definition of $\underline{\mathrm{Ad}}(\mathrm{G})$ *involves* $\overline{Q}(m)$*).*

This proposition implies the theorem, since we now have a prop morphism $\mathrm{COB} \to S(\mathbf{Cob})$, obtained by extending $\overline{Q} : \mathrm{Bialg} \to S(\mathbf{Cob})$ by $R \mapsto \mathrm{G}$.

Let us now prove Proposition 6.10. We start by making (40) explicit: this means that

$$\overline{Q}(m) = (\xi'')^{\boxtimes 2} \circ \overline{Q}(m) \circ (\xi'')^{-1}, \quad (\xi'')^{\boxtimes 2} \circ (21) \circ \overline{Q}(\Delta) \circ (\xi'')^{-1} = \underline{\mathrm{Ad}}(\mathrm{F}') \circ \overline{Q}(\Delta),$$

$\overline{Q}(\eta) = \xi'' \circ \overline{Q}(\eta)$, $\overline{Q}(\varepsilon) = \overline{Q}(\varepsilon) \circ \xi''$.

We first prove:

Lemma 6.11. *There exists a unique* $\mathrm{H} \in \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})^\times$ *such that* $\mathrm{H} = \mathrm{inj}_0^{\otimes 2}$ *+ terms of degree* ≥ 1 *in* ρ*, and* $((\xi'')^{\boxtimes 2} \circ \mathrm{H}) *_{\overline{Q}} \mathrm{H} = ((\xi'')^{\boxtimes 2} \circ (\mathrm{F}')^{2,1}) *_{\overline{Q}} \mathrm{F}'$*. Then* $(\xi'')^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-1} = \underline{\mathrm{Ad}}(\mathrm{H}) \circ \overline{Q}(\Delta)$*,* H *satisfies the identities (41), and* $\mathrm{H} = \mathrm{inj}_0^{\otimes 2}$ *+ terms of degree* ≥ 2 *in* ρ*.*

Proof of Lemma. The existence of H is a consequence of the following statement. Let $A = A^0 \supset A^1 \supset \ldots$ be a filtered algebra, complete and separated for this filtration. Let θ be a topological automorphism of A, such that $(\theta - \mathrm{id}_A)(A^n) \subset A^{n+1}$ for any n. Let $u \in A$ be such that $u \equiv 1$ modulo A^1. Then there exists a unique $v \in A$, with $v \equiv 1$ modulo A^1 and $v\theta(v) = u$. We will apply this statement to $A = \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$ equipped with the product given by $\overline{Q}(m)$. The filtration is given by the degree in ρ, and $\theta(\mathrm{F}) = (\xi'')^{\boxtimes 2} \circ \mathrm{F}$.

To prove the existence of v, we construct inductively the class $[v]_n$ of v in A/A^n: assume that $[v]_n$ has been found such that $[v]_n\theta([v]_n) = [u]_n$ in A/A^n, and let v' be a lift of $[v]_n$ to A/A^{n+1}, then $v'\theta(v') \equiv [u]_{n+1}$ modulo A^n/A^{n+1}. Then we set $[v]_{n+1} = v' - (1/2)(v'\theta(v') - [u_{n+1}])$ (in A/A^{n+1}).

Let us prove the uniqueness of v. Let v and v' be solutions; let us prove by induction on n that $[v]_n = [v']_n$. Assume that this has been proved up to order $n-1$ and let us prove it at order n. We have $v\theta(v) - v'\theta(v') = (v - v')\theta(v) + v'(\theta(v) - \theta(v'))$. Then we have $v - v' \in A^{n-1}$, $\theta(v) - \theta(v') \in A^{n-1}$, and the classes of these elements are equal in A^{n-1}/A^n. So the class of $v\theta(v) - v'\theta(v')$ in A^{n-1}/A^n is equal to twice the class of $v - v'$ in A^{n-1}/A^n. Since $v\theta(v) = v'\theta(v')$, the latter class is 0, so $v - v' \in A^n$.

Before we prove the properties of H, we construct the following propic version of the theory of twists. Let us denote by $\boldsymbol{\Delta}$ the set of all $\overline{\Delta} \in \mathbf{Cob}(S, S^{\otimes 2})$, such that there exists a prop morphism $\overline{Q}_{\overline{\Delta}} : \mathrm{Bialg} \to S(\mathbf{Cob})$, such that $\overline{Q}_{\overline{\Delta}}(\Delta) = \overline{\Delta}$ and $\overline{Q}_{\overline{\Delta}}(m) = \overline{Q}(m)$, $\overline{Q}_{\overline{\Delta}}(\varepsilon) = \overline{Q}(\varepsilon)$, $\overline{Q}_{\overline{\Delta}}(\eta) = \overline{Q}(\eta)$. For $\overline{\Delta}_1 \in \boldsymbol{\Delta}$, we denote by $\mathrm{Tw}(\overline{\Delta}_1, -)$ the set of all $\mathrm{F}_1 \in \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})^\times$ satisfying (41) where the underlying structure is that given by $\overline{Q}_{\overline{\Delta}_1}$. Then if $\overline{\Delta}_2 := \mathrm{Ad}(\mathrm{F}_1) \circ \overline{\Delta}_1$, we have $\overline{\Delta}_2 \in \boldsymbol{\Delta}$. If $\overline{\Delta}_1, \overline{\Delta}_2 \in \boldsymbol{\Delta}$, let us denote by $\mathrm{Tw}(\overline{\Delta}_1, \overline{\Delta}_2) \subset \mathrm{Tw}(\overline{\Delta}_1, -)$ the set of all F_1 such that $\overline{\Delta}_2 = \underline{\mathrm{Ad}}(\mathrm{F}_1) \circ \overline{\Delta}_1$.

Then if $\overline{\Delta}_i \in \boldsymbol{\Delta}$ $(i = 1, 2, 3)$, the map $(\mathrm{F}_1, \mathrm{F}_2) \mapsto \mathrm{F}_2\,\mathrm{F}_1$ (product in $\mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$ using $\overline{Q}(m)$) defines a map $\mathrm{Tw}(\overline{\Delta}_1, \overline{\Delta}_2) \times \mathrm{Tw}(\overline{\Delta}_2, \overline{\Delta}_3) \to \mathrm{Tw}(\overline{\Delta}_1, \overline{\Delta}_3)$.

Let us now prove the properties of H. We have $\mathrm{F}' \in \mathrm{Tw}(\overline{Q}(\Delta), (\xi'')^{\boxtimes 2} \circ (21) \circ \overline{Q}(\Delta) \circ (\xi'')^{-1})$ and $(\xi'')^{\boxtimes 2} \circ (\mathrm{F}')^{2,1} \in \mathrm{Tw}\left((\xi'')^{\boxtimes 2} \circ (21) \circ \overline{Q}(\Delta) \circ (\xi'')^{-1}, ((\xi'')^2)^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-2}\right)$, therefore

$$\mathcal{F} := \left((\xi'')^{\boxtimes 2} \circ (\mathrm{F}')^{2,1}\right) *_{\overline{Q}} \mathrm{F}' \in \mathrm{Tw}\left(\overline{Q}(\Delta), ((\xi'')^2)^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-2}\right).$$

In particular, we have

$$((\xi'')^2)^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-2} = \underline{\mathrm{Ad}}(\mathcal{F}) \circ \overline{Q}(\Delta).$$

Then if we set

$$\mathcal{F}(n) = ((\xi'')^{\boxtimes 2(n-1)} \circ \mathcal{F}) *_{\overline{Q}} \dots *_{\overline{Q}} ((\xi'')^{\boxtimes 2} \circ \mathcal{F}) *_{\overline{Q}} \mathcal{F},$$

we have for n integer ≥ 0,

$$((\xi'')^{2n})^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-2n} = \underline{\mathrm{Ad}}(\mathcal{F}(n)) \circ \overline{Q}(\Delta). \tag{43}$$

$\mathbf{Cob}(S, S)$ is the completion of a $\mathbb{N}$-graded algebra, where the degree is given by $\deg(\rho) = 1$, $\deg(\mu) = 0$. $\xi'' \in \mathbf{Cob}(S, S)$ is equal to the identity modulo terms of positive degree. We have therefore a unique formal map $t \mapsto (\xi'')^t$, inducing a polynomial map $\mathbf{k} \to \mathbf{Cob}(S, S)/\{$its part of degree $> k\}$ for each $k \geq 0$, which coincides with the map $n \mapsto$ (class of $(\xi'')^n$) for $t \in \mathbb{N}$.

On the other hand, one checks that there is a unique formal map $t \mapsto \mathcal{F}(t)$ with values in $\mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$, such that the induced map $\mathbf{k} \to \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})/\{$its part of degree $> k\}$ is polynomial for any $k \geq 0$ and coincides with the maps $n \mapsto$ (class of $\mathcal{F}(n)$) for $t \in \mathbb{N}$.

It follows that (43) also holds when n is replaced by the formal variable t. The resulting identity can be specialized for $n = 1/2$. The specialization of $(\xi'')^{2t}$ for $t = 1/2$ is ξ''.

We now prove that $\mathcal{F}(1/2) = \mathrm{H}$.

Let us set $\mathrm{H}(n) = ((\xi'')^{\boxtimes 2(n-1)} \circ \mathrm{H}) *_{\overline{Q}} \dots *_{\overline{Q}} ((\xi'')^{\boxtimes 2} \circ \mathrm{H}) *_{\overline{Q}} \mathrm{H}$. Then we have a unique formal map $t \mapsto \mathrm{H}(t)$ with values in $\mathbf{Cob}(\mathbf{1}, S^{\otimes 2})$, such that the induced map $\mathbf{k} \to \mathbf{Cob}(\mathbf{1}, S^{\otimes 2})/\{$its part of degree $> k\}$ is polynomial for any $k \geq 0$ and coincides with the maps $n \mapsto$ (class of $\mathrm{H}(n)$) for $t \in \mathbb{N}$.

We have $\mathrm{H}(2n) = \mathcal{F}(n)$ for any integer n, so this identity also holds when n is replaced by the formal variable t. Specializing the resulting identity for $t = 1/2$, we get $\mathcal{F}(1/2) = \mathrm{H}(1) = \mathrm{H}$.

The specialization of the formal version of (43) for $t = 1/2$ then gives

$$(\xi'')^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-1} = \underline{\mathrm{Ad}}(\mathrm{H}) \circ \overline{Q}(\Delta). \tag{44}$$

Let us now prove the identities (41) in H. We have $(\xi'' \circ (\mathrm{H}^{\emptyset,1})) *_{\overline{Q}} \mathrm{H}^{\emptyset,1} = \mathrm{inj}_0$ and $\mathrm{H}^{\emptyset,1} = \mathrm{inj}_0$ + terms of positive degree in ρ, hence by the uniqueness result proved above $\mathrm{H}^{\emptyset,1} = \overline{Q}(\eta)$. In the same way, $\mathrm{H}^{1,\emptyset} = \mathrm{inj}_0$.

We now prove that

$$(\mathrm{H})_{\overline{Q}}^{1,2} *_{\overline{Q}} (\mathrm{H})_{\overline{Q}}^{12,3} = (\mathrm{H})_{\overline{Q}}^{2,3} *_{\overline{Q}} (\mathrm{H})_{\overline{Q}}^{1,23}. \tag{45}$$

Let $\Psi \in \mathbf{Cob}(\mathbf{1}, S^{\otimes 3})^\times$ be such that $(\mathrm{H})_{\overline{Q}}^{1,2} *_{\overline{Q}} (\mathrm{H})_{\overline{Q}}^{12,3} = (\mathrm{H})_{\overline{Q}}^{2,3} *_{\overline{Q}} (\mathrm{H})_{\overline{Q}}^{1,23} *_{\overline{Q}} \Psi$.

We have $(\mathcal{F})_{\overline{Q}}^{1,2} *_{\overline{Q}} (\mathcal{F})_{\overline{Q}}^{12,3} = (\mathcal{F})_{\overline{Q}}^{2,3} *_{\overline{Q}} (\mathcal{F})_{\overline{Q}}^{1,23}$, that is

$$(\mathcal{F} \boxtimes \overline{Q}(\eta)) *_{\overline{Q}} (\overline{Q}(\Delta \boxtimes \eta) \circ \mathcal{F}) = (\overline{Q}(\eta) \boxtimes \mathcal{F}) *_{\overline{Q}} (\overline{Q}(\eta \boxtimes \Delta) \circ \mathcal{F})$$

This is rewritten

$$\begin{aligned} &\left((\xi'')^{\boxtimes 3} \circ (\mathrm{H} \boxtimes \overline{Q}(\eta))\right) *_{\overline{Q}} (\mathrm{H} \boxtimes \overline{Q}(\eta)) *_{\overline{Q}} (\overline{Q}(\Delta \boxtimes \eta) \circ \xi''^{\boxtimes 2} \circ \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\Delta \boxtimes \eta) \circ \mathrm{H}) \\ &= \left((\xi'')^{\boxtimes 3} \circ (\overline{Q}(\eta) \boxtimes \mathrm{H})\right) *_{\overline{Q}} (\overline{Q}(\eta) \boxtimes \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\eta \boxtimes \Delta) \circ \xi''^{\boxtimes 2} \circ \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\eta \boxtimes \Delta) \circ \mathrm{H}) \end{aligned}$$

Using $\xi'' \circ \mathrm{inj}_0 = \mathrm{inj}_0$ and (44), we get

$$\begin{aligned} &\left((\xi'')^{\boxtimes 3} \circ (\mathrm{H} \boxtimes \overline{Q}(\eta))\right) *_{\overline{Q}} (\xi''^{\boxtimes 3} \circ \overline{Q}(\Delta \boxtimes \eta) \circ \mathrm{H}) *_{\overline{Q}} (\mathrm{H} \boxtimes \overline{Q}(\eta)) *_{\overline{Q}} (\overline{Q}(\Delta \boxtimes \eta) \circ \mathrm{H}) \\ &= \left((\xi'')^{\boxtimes 3} \circ (\overline{Q}(\eta) \boxtimes \mathrm{H})\right) *_{\overline{Q}} (\xi''^{\boxtimes 3} \circ \overline{Q}(\eta \boxtimes \Delta) \circ \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\eta) \boxtimes \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\eta \boxtimes \Delta) \circ \mathrm{H}) \end{aligned}$$

and since $X \mapsto (\xi'')^{\boxtimes 3} \circ X$ is an automorphism of $\mathbf{Cob}(\mathbf{1}, S^{\otimes 3})$, we get

$$\Big((\xi'')^{\boxtimes 3} \circ \Big((\mathrm{H} \boxtimes \overline{Q}(\eta))(\overline{Q}(\Delta \boxtimes \eta) \circ \mathrm{H})\Big)\Big) *_{\overline{Q}} (\mathrm{H} \boxtimes \overline{Q}(\eta)) *_{\overline{Q}} (\overline{Q}(\Delta \boxtimes \eta) \circ \mathrm{H})$$
$$= \Big((\xi'')^{\boxtimes 3} \circ \Big((\overline{Q}(\eta) \boxtimes \mathrm{H})(\overline{Q}(\eta \boxtimes \Delta) \circ \mathrm{H})\Big)\Big) *_{\overline{Q}} (\overline{Q}(\eta) \boxtimes \mathrm{H}) *_{\overline{Q}} (\overline{Q}(\eta \boxtimes \Delta) \circ \mathrm{H}),$$

i.e.,

$$((\xi'')^{\boxtimes 3} \circ \Psi) *_{\overline{Q}} \big((\mathrm{H})^{2,3}_{\overline{Q}} *_{\overline{Q}} (\mathrm{H})^{1,23}_{\overline{Q}}\big) *_{\overline{Q}} \Psi = ((\mathrm{H})^{2,3}_{\overline{Q}} *_{\overline{Q}} (\mathrm{H})^{1,23}_{\overline{Q}}).$$

We now prove that this implies that $\Psi = \mathrm{inj}_0^{\boxtimes 3}$.

For this, we apply the following general statement. Let $A = A^0 \supset A^1 \supset ...$ be an algebra equipped with a decreasing filtration, complete and separated for this filtration. Let θ be a topological automorphism of A, such that $(\theta - \mathrm{id}_A)(A^n) \subset A^{n+1}$. Let $X \in A$ be such that $X \equiv 1$ modulo A^1. Let $x \in A$ be such that $x \equiv 1$ modulo A^1, and

$$\theta(x)Xx = X.$$

Then $x = 1$. This is proved by induction. Assume that we have proved that $x \equiv 1$ modulo A^{n-1}. Then $\theta(x)XxX^{-1} \equiv 1 + 2(x-1)$ modulo A^n. Therefore $x \equiv 1$ modulo A^n. Finally $x = 1$.

We then apply the general statement to $A = \mathbf{Cob}(\mathbf{1}, S^{\otimes 3})$ and $\theta : X \mapsto (\xi'')^{\boxtimes 3} \circ X$ and get $\Psi = \mathrm{inj}_0^{\boxtimes 3}$. This implies (45). This ends the proof of Lemma 6.11. □

We now end the proof of Proposition 6.10. Lemma 6.11 says that $\mathrm{H} \in \mathrm{Tw}(\overline{Q}(\Delta), (\xi'')^{\boxtimes 2} \circ \overline{Q}(\Delta) \circ (\xi'')^{-1})$, and since $\mathrm{F}' \in \mathrm{Tw}(\overline{Q}(\Delta), (\xi'')^{\boxtimes 2} \circ (21) \circ \overline{Q}(\Delta) \circ (\xi'')^{-1})$, we have

$$\mathrm{G}' := \big((21) \circ \mathrm{H}^{-1}\big) *_{\overline{Q}} \mathrm{F}' \in \mathrm{Tw}(\overline{Q}(\Delta), (21) \circ \overline{Q}(\Delta)).$$

Let us set $\mathcal{G}' := (\mathrm{G}')^{2,1} *_{\overline{Q}} \mathrm{G}'$, then $\mathcal{G}' \in \mathrm{Tw}(\overline{Q}(\Delta), \overline{Q}(\Delta))$. For any integer $n \geq 0$, we then have $(\mathcal{G}')^n \in \mathrm{Tw}(\overline{Q}(\Delta), \overline{Q}(\Delta))$. As before, there exists a unique formal map $t \mapsto (\mathcal{G}')^t$, such that the map $t \mapsto$ (class of $(\mathcal{G}')^t$) in $\mathbf{Cob}(\mathbf{1}, S^{\otimes n})/\{$its part of degree $\geq k\}$ is polynomial and extends $n \mapsto (\mathcal{G}')^n$. Specializing for $t = -1/2$, we get $(\mathcal{G}')^{-1/2} \in \mathrm{Tw}(\overline{Q}(\Delta), \overline{Q}(\Delta))$. Set

$$\mathrm{G} := \mathrm{G}' *_{\overline{Q}} (\mathcal{G}')^{-1/2} = \mathrm{G}' *_{\overline{Q}} (\mathrm{G}'^{2,1} *_{\overline{Q}} \mathrm{G}')^{-1/2},$$

then

$$\mathrm{G} \in \mathrm{Tw}(\overline{Q}(\Delta), (21) \circ \overline{Q}(\Delta)).$$

Then we have: $\mathrm{G}' *_{\overline{Q}} (\mathrm{G}'^{2,1} *_{\overline{Q}} \mathrm{G}')^n = (\mathrm{G}' *_{\overline{Q}} \mathrm{G}'^{2,1})^n *_{\overline{Q}} \mathrm{G}'$ for any integer $n \geq 0$, so this identity also holds when n is replaced by a formal variable t. Specializing the latter identity to $t = 1/2$, we get $\mathrm{G} = (\mathrm{G}' *_{\overline{Q}} \mathrm{G}'^{2,1})^{-1/2} *_{\overline{Q}} \mathrm{G}'$. Then $\mathrm{G} *_{\overline{Q}} \mathrm{G}^{2,1} = \mathrm{G}' *_{\overline{Q}} (\mathrm{G}'^{2,1} *_{\overline{Q}} \mathrm{G}')^{-1} *_{\overline{Q}} \mathrm{G}'^{2,1} = \mathrm{inj}_0^{\boxtimes 2}$, so we also have $\mathrm{G}^{2,1} *_{\overline{Q}} \mathrm{G} = \mathrm{inj}_0^{\boxtimes 2}$.

This ends the proof of Proposition 6.10, and therefore also of Theorem 6.8. □

Remark 6.12. The proof of Proposition 6.10 is a propic version of the proof of the following statement. Let (U, m_U, η_U) be a formal deformation over $\mathbf{k}[[\hbar]]$ of an enveloping algebra $U(\mathfrak{a})$ (as an algebra). Let $\mathbf{\Delta}_U$ be the set of all morphisms $\Delta : U \to U^{\otimes 2}$ such that $(U, m_U, \Delta, \eta_U, \varepsilon_U)$ is a QUE algebra formally deforming the bialgebra $U(\mathfrak{a})$. If $\Delta_1, \Delta_2 \in \mathbf{\Delta}_U$, we say that $F_U \in \mathrm{Tw}(\Delta_1, \Delta_2)$ iff $F_U \in (U^{\otimes 2})^\times$, $(\varepsilon_U \otimes \mathrm{id}_U)(F_U) = (\mathrm{id}_U \otimes \varepsilon_U)(F_U) = 1_U$, $(F_U \otimes 1_U)(\Delta_1 \otimes \mathrm{id}_U)(F_U) = (1_U \otimes F_U)(\mathrm{id}_U \otimes \Delta_1)(F_U)$ and $\Delta_2 = \mathrm{Ad}(F_U) \circ \Delta_1$, where $\mathrm{Ad}(F_U) : U^{\otimes 2} \to U^{\otimes 2}$ is given by $x \mapsto F_U x F_U^{-1}$ (and $1_U = \eta_U(1)$). For $\Delta \in \mathbf{\Delta}_U$ and $\theta_U \in \mathrm{Aut}(U, m_U, \eta_U)$ such that $\theta_U = \mathrm{id}_U + O(\hbar)$, we also have $\Delta^{21} \in \mathbf{\Delta}_U, \theta_U^{\otimes 2} \circ \Delta^{21} \circ \theta_U^{-1} \in \mathbf{\Delta}_U$. The statement is that if for such Δ, θ_U, there exists $F_U \in \mathrm{Tw}(\Delta, \theta_U^{\otimes 2} \circ \Delta^{21} \circ \theta_U^{-1})$, then there exists $G_U \in \mathrm{Tw}(\Delta, \Delta^{21})$ such that $G_U G_U^{2,1} = 1_U^{\otimes 2}$.

6.4. **Relation with quasi-Poisson manifolds.** Define a coboundary quasi-Lie bialgebra (QLBA) as a set $(\mathfrak{g}, \mu_{\mathfrak{g}}, \delta_{\mathfrak{g}}, Z_{\mathfrak{g}}, r_{\mathfrak{g}})$, where $(\mathfrak{g}, \mu_{\mathfrak{g}}, \delta_{\mathfrak{g}}, Z_{\mathfrak{g}})$ is a quasi-Lie bialgebra, and $r_{\mathfrak{g}} \in \wedge^2(\mathfrak{g})$ is such that $\delta_{\mathfrak{g}}(x) = [r_{\mathfrak{g}}, x \otimes 1 + 1 \otimes x]$. In [Dr2], coboundary QUE quasi-Hopf algebras were introduced; the classical limit of this structure is a coboundary QLBA.

According to [Dr2], Proposition 3.13, a coboundary QUE quasi-Hopf algebra with classical limit the coboundary QLBA $(\mathfrak{g}, \mu_{\mathfrak{g}}, \delta_{\mathfrak{g}}, Z_{\mathfrak{g}}, r_{\mathfrak{g}})$ is twist-equivalent to a coboundary QUE Hopf algebra of the form $(U(\mathfrak{g}_\hbar), m_0, \Delta_0, R = 1, \Phi = \mathcal{E}(\hbar^2 Z_\hbar))$, where $\mathfrak{g}_\hbar$ is a deformation of $\mathfrak{g}$ (as a Lie algebra) in the category of topologically free $\mathbf{k}[[\hbar]]$-modules, $Z_\hbar \in \wedge^3(\mathfrak{g}_\hbar)^{\mathfrak{g}_\hbar}$ is a deformation of $Z_{\mathfrak{g}} + (\delta_{\mathfrak{g}} \otimes \mathrm{id})(r_{\mathfrak{g}}) + \mathrm{c.\,p.} - \mathrm{CYB}(r_{\mathfrak{g}})$, and $\mathcal{E}(Z) = 1 + Z/6 + ...$ is a series introduced in [Dr2] (m_0, Δ_0 are the undeformed operations).

Let now $(\mathfrak{a}, r_{\mathfrak{a}})$ be a coboundary Lie bialgebra. Let $(U_\hbar(\mathfrak{a}), R_{\mathfrak{a}})$ be a quantization of it: this is a coboundary QUE Hopf algebra. Applying to it the above result, we obtain:

a) there exists a deformation $\mathfrak{a}_\hbar$ of $\mathfrak{a}$ in the category of topologically free $\mathbf{k}[[\hbar]]$-Lie algebras, such that $U_\hbar(\mathfrak{a})$ is isomorphic to $U(\mathfrak{a}_\hbar)$ as an algebra;

b) there exists $J \in U(\mathfrak{a}_\hbar)^{\otimes 2}$ of the form $J = 1 + \hbar r_{\mathfrak{a}}/2 + O(\hbar^2)$ such that $J^{2,3} J^{1,23} \mathcal{E}(\hbar^2 Z_\hbar) = J^{1,2} J^{12,3}$, where $Z_\hbar \in \wedge^3(\mathfrak{a}_\hbar)^{\mathfrak{a}_\hbar}$ is a deformation of $Z_{\mathfrak{a}}$.

If A is a Lie group with Lie algebra $\mathfrak{a}$, then $r_{\mathfrak{a}}$ induces a quasi-Poisson homogeneous structure on A under the action of the quasi-Lie bialgebra $(\mathfrak{a}, \delta_{\mathfrak{a}} = 0, Z_{\mathfrak{a}})$: the action of $\mathfrak{a}$ is the regular left action, and the quasi-Poisson structure is $\{f, g\} = m \circ \mathbf{L}^{\otimes 2}(r_{\mathfrak{a}})(f \otimes g)$, where f, g are functions on A (and m is the product of functions). As explained in [EE], J constructed above gives rise to a quantization of this quasi-Poisson homogeneous space, compatible with the quasi-Hopf algebra $(U(\mathfrak{a}_\hbar), m_0, \Delta_0, R = 1, \Phi = \mathcal{E}(\hbar^2 Z_\hbar))$.

Notice that the deformation class of $(\mathfrak{a}_\hbar, Z_\hbar)$ is a priori dictated by $r_{\mathfrak{a}}$.

References

[BN] D. Bar-Natan, *On associators and the Grothendieck-Teichmüller group, I,* Selecta Math. (N.S.) **4** (1998), no. 2, 183–212.

[Bou] N. Bourbaki, Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Actualités Scientifiques et Industrielles, No. 1349. Hermann, Paris, 1972.

[DS] P. Dazord, D. Sondaz, *Groupes de Poisson affines,* Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), 99–128, Math. Sci. Res. Inst. Publ., 20, Springer, New York, 1991.

[Dr1] V. Drinfeld, *Quantum groups,* Proceedings of the ICM-86, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[Dr2] V. Drinfeld, *Quasi-Hopf algebras,* Leningrad Math. J. **1** (1990), no. 6, 1419–1457.

[Dr3] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991), no. 4, 829–860.

[Dr4] V. Drinfeld, *On some unsolved problems in quantum group theory,* Quantum groups (Leningrad, 1990), 1–8, Lecture Notes in Math., **1510,** Springer, Berlin, 1992.

[Dr5] V. Drinfeld, *On Poisson homogeneous spaces of Poisson-Lie groups,* Theoret. and Math. Phys. 95 (1993), no. 2, 524–525.

[Enr1] B. Enriquez, *Quantization of Lie bialgebras and shuffle algebras of Lie algebras,* Selecta Math. (N.S.) **7** (2001), no. 3, 321–407.

[Enr2] B. Enriquez, *On some universal algebras associated to the category of Lie bialgebras,* Advances in Mathematics **164** (2001), 1–23.

[Enr3] B. Enriquez, *A cohomological construction of quantization functors of Lie bialgebras,* Advances in Mathematics **197** (2005) 430–479.

[EE] B. Enriquez, P. Etingof, *Quantization of Alekseev-Meinrenken dynamical* r*-matrices,* Lie groups and symmetric spaces, 81–98, Amer. Math. Soc. Transl. Ser. 2, 210, Amer. Math. Soc., Providence, RI, 2003.

[EK1] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras, I,* Selecta Math. (N.S.) **2** (1996), no. 1, 1–41.

[EK2] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras, II,* Selecta Math. (N.S.) **4** (1998), no. 2, 213–231.

[H] G. Halbout, *Formality theorem for Lie bialgebras and quantization of coboundary r-matrices,* math.QA/0506487, to appear in Advances in Mathematics.

[McL] S. McLane, *Categorical algebra,* Bull. Amer. Math. Soc. **71** (1965), 40–106.

[Pos] L. Positselski, letter to M. Finkelberg and R. Bezrukavnikov, 1995 (in Russian).

[KPST] S. Khoroshkin, I. Pop, A. Stolin, V. Tolstoy, *On some Lie bialgebra structures on polynomial algebras and their quantization,* preprint of the Mittag-Leffler Institute, no. 21, 2003/2004.

[Tam] D. Tamarkin, *Quantization of Lie bialgebras via the formality of the operad of little disks,* Deformation quantization (Strasbourg, 2001), 203–236, IRMA Lect. Math. Theor. Phys., **1,** de Gruyter, Berlin, 2002.

IRMA (CNRS), rue René Descartes, F-67084 Strasbourg, France
E-mail address: enriquez@math.u-strasbg.fr

IRMA (CNRS), rue René Descartes, F-67084 Strasbourg, France
E-mail address: halbout@math.u-strasbg.fr